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FIXED POINTS OF COMPATIBLE MAPPINGS SATISFYING THE MANN TYPE ITERATION

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Abstract In this paper, we give some fixed point theorems for compatible mappings satisfying the Mann type iteration Our results extend and improve some theorems of Abbaoui, Rhoades and others

Let F and S be two mappings a normed linear space $(X, \|\cdot\|)$ into itself. In [4], Sessa defined the mappings F and S to be weakly commuting if

$$\|FSx - SFx\| \leq \|Fx - Sx\|$$

for all $x \in X$. Clearly, any two commuting mappings are weakly commutings, but the converse is not true ([4]). Recently, Jungck [2] generalized the concept of weakly commuting mappings in the following way, i.e., the mappings F and S are said to be compatible if

$$\lim_{n\to\infty} \|FSx_n - SFx_n\| = 0$$

when $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Fx_n = \lim_{n\to\infty} Sx_n = z$ for some $z \in X$.

It is obvious that any two weakly commuting mappings are compatible, but the converse is not true. Some examples for this fact can be found in [2].

Let R^+ denote the set of all nonnegative real numbers

We need the following lemma for our main theorems:

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LEMMA 1. *Let F and S be compatible mappings from a normed linear space $(X, \|\cdot\|)$ into itself. If $Fz = Sz$ for some $z \in X$, then $FFz = SFz = FSz = SSz$.*

Recently, Abbaoui [1] proved the following theorem:

THEOREM A. *Let $(X, \|\cdot\|)$ be a normed linear space and C be a nonempty closed, convex, bounded subset of X. Let $\Phi : R^+ \to R^+$ be a nondecreasing function such that $\Phi(t+) < t$ for all $t \in R^+$. Let F and S be mappings from C into itself satisfying the following conditions:*
(1) *The pair $\{F, S\}$ is weakly commuting,*
(2) *$F^2 = S^2 = I$ (: the identity mapping),*

$$(3)\qquad \begin{aligned} \|Fx - Fy\|^2 \le \Phi\big(\max\{&\|Fx - Sx\|\|Fy - Sy\|, \\ &\|Fy - Sx\|\|Fx - Sy\|, \\ &\|Fx - Sx\|\|Fx - Sy\|, \\ &\|Fy - Sx\|\|Fy - Sy\|\}\big) \end{aligned}$$

for all $x, y \in X$. Let x_1 be an arbitrary point of C and, for $c_n \in (0, 1)$, define

$$(4)\qquad Sx_{n+1} = (1 - c_n)Sx_n + c_nFx_n$$

for $n = 1, 2 \cdots$. If the sequence $\{Sx_n\}$ converges to a point $p \in C$, then $\{Fx_n\}$ also converges to the point p and p is the unique common fixed point of F and S.

In [3], Rhoades generalized Theorem A as follows:

THEOREM B. *Let $(X, \|\cdot\|)$ be a normed linear space and C be a nonempty closed, convex, bounded subset of X. Let $\Phi : R^+ \to R^+$ be a nondecreasing function such that $\Phi(t+) < t$ for all $t \in R^+$. Let F and S be mappings from C into itself satisfying (2) and the following conditions:*
(5) *The pair $\{F, S\}$ is compatible,*

$$(6)\qquad \begin{aligned} \|Fx - Fy\|^2 \le \Phi(\max\{&\|Sx - Sy\|^2, \|Fx - Sx\|^2, \\ &\|Fy - Sy\|^2, \|Fy - Sx\|^2, \|Fx - Sy\|^2\}) \end{aligned}$$

for all $x, y \in C$. Let x_0 be an arbitrary point of C and $\{c_n\}$ be a real sequence such that

(a) $c_0 = 1$,

(b) $0 \le c_n < 1$ *for* $n = 1, 2, \cdots$,

(c) $\liminf_{n\to\infty} c_n = \alpha > 0$.

If the sequence $\{Sx_n\}$ defined by (4) converges to a point $p \in C$, then $\{Fx_n\}$ also converges to the point p and p is the unique common fixed point of F and S.

REMARK. [3] (i) For all $a, b \in R^+$, $ab \le \max\{a^2, b^2\}$ is always true and so, employing this idea to each of the remaining terms on the right hand side of (3) give rise to the following inequality:

$$\|Fx - Fy\|^2 \le \Phi\big(\max\{\|Gx - Fx\|^2, \|Gy - Fy\|^2, \|Gy - Fx\|^2, \|Gx - Fy\|^2\}\big). \tag{7}$$

Thus the contractive condition (6) is more general than (7).

(ii) Taking the square root of both sides of (7), $\Phi(t) < t$ for all $t > 0$ does not imply that $(\Phi(t))^{1/2} < t$ for all $t > 0$. A simple counterexample is provided by $\Phi(t) = t/(1+t)$ for all $t > 0$.

(iii) Theorem A is clearly a special case of Theorem B.

In this paper, motivated by Theorem B, we prove some common fixed point theorems for compatible mappings satisfying the Mann type iterations. Our main results extend and improve Theorems A, B and others.

Now, we give our main theorems:

THEOREM 2. *Let $(X, \|\cdot\|)$ be a normed linear space and C be a nonempty closed, convex, bounded subset of X. Let $\Phi : R^+ \to R^+$ be a nondecreasing function such that $\Phi(t+) < t$ for all $t \in R^+$. Let F, G, S and T be four mappings from C into itself satisfying the following conditions:*

(8) *The pairs $\{F, S\}$ and $\{G, T\}$ are compatible,*

$$\|Sx - Ty\|^2 \le \Phi\big(\max\{\|Fx - Gy\|^2, \|Fx - Sx\|^2, \|Gy - Ty\|^2, \|Gy - Sx\|^2, \|Fx - Ty\|^2\}\big) \tag{9}$$

for all $x, y \in C$. Let $\{c_n\}$ be a real sequence satisfying the conditions (a), (b) *and* (c) *in Theorem B. For an arbitrary point $x_0 \in X$, define a sequence $\{x_n\}$ in X by*

$$\begin{cases} Fx_{2n+1} = (1 - c_n)Fx_{2n} + c_n S x_{2n}, \\ Gx_{2n+2} = (1 - c_n)Gx_{2n+1} + c_n T x_{2n+1} \end{cases} \tag{10}$$

for $n = 0, 1, 2, \cdots$. If the sequence $\{x_n\}$ defined by (10) *converges to a point $z \in C$ and if F and G are continuous at the point z, then Tz is a unique common fixed point of F, G, S and T.*

Proof. First of all, we prove that $Fz = Gz = Sz = Tz$. From (10), it follows that

$$c_n S x_{2_n} = F x_{2n+1} - (1 - c_n) F x_{2n}. \tag{11}$$

Since F is continuous at z, from (11), we have

$$\lim_{n\to\infty} Sx_{2n} = \lim_{n\to\infty} Fx_n = Fz.$$

Similarly, from (10), it follows that

$$\lim_{n\to\infty} Tx_{2n+1} = \lim_{n\to\infty} Gx_n = Gz$$

On the other hand, by (9), we have

$$\begin{aligned} \|Sx_{2n} - Tz\| \le \Phi(\max\{&\|Fx_{2n} - Gz\|^2, \|Fx_{2n} - Sx_{2n}\|^2, \\ &\|Gz - Tz\|^2, \|Gz - Sx_{2n}\|^2, \|Fx_{2n} - Tz\|^2\}). \end{aligned} \tag{12}$$

Letting $n \to \infty$ in (12), we have

$$\begin{aligned} &\|Fz - Tz\|^2 \\ &\le \Phi(\max\{\|Fz - Gz\|^2, \|Fz - Fz\|^2, \\ &\qquad \|Gz - Tz\|^2, \|Gz - Fz\|^2, \|Fz - Tz\|^2\}), \\ &= \Phi(\max\{\|Fz - Gz\|^2, \|Fz - Tz\|^2, \|Gz - Tz\|^2\}). \end{aligned} \tag{13}$$

Similarly, we can show that

$$(14)\quad \|Gz - Sz\|^2 \le \Phi(\max\{\|Gz - Fz\|^2, \|Gz - Sz\|^2, \|Fz - Sz\|^2\}).$$

If $\|Fz-Tz\|^2 = \max\{\|Fz-Gz\|^2, \|Fz-Tz\|^2, \|Gz-Tz\|^2\}$ in (13), we have

$$\|Fz - Tz\|^2 \le \Phi(\|Fz - Tz\|^2) < \|Fz - Tz\|^2,$$

which is a contradiction and so

$$(15)\qquad \|Fz - Tz\|^2 \le \Phi(\max\{\|Fz - Gz\|^2, \|Gz - Tz\|^2\}).$$

Similarly, from (14), it follows that

$$(16)\qquad \|Gz - Sz\|^2 \le \Phi(\max\{\|Fz - Gz\|^2, \|Fz - Sz\|^2\}).$$

Using (9) again, we have

$$(17)\qquad \begin{aligned} &\|Sx_{2n} - Tx_{2n+1}|^2\| \\ &\le \Phi(\max\{\|Fx_{2n} - Gx_{2n+1}\|^2, \|Fx_{2n} - Sx_{2n}\|^2, \\ &\qquad \|Gx_{2n+1} - Tx_{2n+1}\|^2, \|Gx_{2n+1} - Sx_{2n}\|^2, \\ &\qquad \|Fx_{2n} - Tx_{2n+1}\|^2\}). \end{aligned}$$

Letting $n \to \infty$ in (17) and $Fz \neq Gz$, we have

$$\begin{aligned} \|Fz - Gz\|^2 &\le \Phi(\max\{\|Fz - Gz\|^2, \|Fz - Fz\|^2, \\ &\qquad \|Gz - Gz\|^2, \|Fz - Gz\|^2, \|Fz - Gz\|^2\}) \\ &= \Phi(\|Fz - Gz\|^2) < \|Fz - Gz\|^2, \end{aligned}$$

which is a contradiction and so we have $Fz = Gz$. Thus, from (15) and (16), it follows that, if $Fz \neq Tz$ and $Gz \neq Sz$, then

$$\|Fz - Tz\|^2 \le \Phi(\|Gz - Tz\|^2) < \|Gz - Tz\|^2 = \|Fz - Tz\|^2,$$
$$\|Gz - Tz\|^2 \le \Phi(\|Fz - Sz\|^2) < \|Fz - Sz\|^2 = \|Gz - Sz\|^2,$$

respectively, which are contradictions and so

$$Fz = Gz = Sz = Tz. \tag{18}$$

Thus, since the pair $\{F, S\}$ is compatible and $Fz = Sz$, by Lemma 1, we have

$$FSz = SFz. \tag{19}$$

By (9), (18) and (19), if $STz \neq Tz$, then we have

$$\begin{aligned}
&\|STz - Tz\| \\
&= \|SSz - Tz\| \\
&\leq \Phi(\max\{\|FSz - Gz\|^2, \|FSz - SSz\|^2, \|Gz - Tz\|^2, \\
&\qquad \|Gz - SSz\|^2, \|FSz - Tz\|^2\}) \\
&\leq \Phi(\max\{[\|FSz - SFz\| + \|SFz - Gz\|]^2, \|FSz - SFz\|^2, \\
&\qquad \|Gz - Gz\|^2, [\|FSz - SFz\| + \|SFz - Tz\|]^2\}) \\
&= \Phi(\max\{\|STz - Tz\|^2, 0, 0, \|STz - Tz\|^2, \|STz - Tz\|^2\}) \\
&= \Phi(\|STz - Tz\|^2) < \|STz - Tz\|^2,
\end{aligned}$$

which is a contradiction and so $STz = Tz$. Since F and S and compatible and $Fz = Sz$, by Lemma 1, we have

$$SFz = SSz = FFz = FTz = STz = Tz$$

and so Tz is a common fixed point of F and S. On the other hand, by interchanging the roles of the pairs $\{F,\ S\}$ and $\{G,\ T\}$ and using (9) again, we have also

$$TTz = GTz = Tz$$

and so Tz is also a common fixed point of G and T. Thus, combining the ahove results, Tz is a common fixed point of F, G, S and T.

The uniqueness of the common fixed point Tz follows fron (9). This completes the proof.

In Thearem 2, if we replace the condition (8) by the following condition:

$$F^2 = S^2 = I \quad \text{or} \quad G^2 = T^2 = I, \tag{20}$$

then Theorem 2 is still true as follows:

THEOREM 3. *Let $(X, \|\cdot\|)$ be a normed linear space and C be a nonempty closed, convex, bounded subset of X. Let $\Phi : R^+ \to R^+$ be a nondecreasing function such that $\Phi(t+) < t$ for all $t \in R^+$. Let F, G, S and T be four mappings from C into itself satisfying the conditions* (9) *and* (20). *Let $\{c_n\}$ be a real sequence satisfying* (a), (b) *and* (c) *in Theorem B. If the sequence $\{x_n\}$ defined by* (10) *converges to a point $z \in C$ and if F and G are continuous at the point z, then z is a unique common fixed point of F, G, S and T.*

Proof. As in the proof of Theorem 2, we have (13), i.e., $Fz = Gz = Sz = Tz$. Suppose that $F^2 = S^2 = I$. If $Tz \neq z$, it follows from (9) and (13) that

$$\begin{aligned} \|z - Tz\|^2 &= \|SSz - Tz\|^2 \\ &\leq \Phi(\max\{\|FSz - Gz\|^2, \|FSz - SSz\|^2, \|Gz - Tz\|^2, \\ &\qquad \|Gz - SSz\|^2, \|FSz - Tz\|^2\}), \\ &= \Phi(\max\{\|F^2z - Tz\|^2, \|F^2z - Sz\|^2, 0, \\ &\qquad \|Tz - S^2z\|^2, \|F^2z - Tz\|^2\}) \\ &\leq \Phi(\|z - Tz\|^2) < \|z - Tz\|^2, \end{aligned}$$

which is a contradiction and so $Tz = z$. Therefore, from (13), it follows that the point z is a common fixed point of E, G, S and T. Similarly, in the case of $G^2 = T^2 = I$, we have $Sz = z$ and so, from (13), the point z is a common fixed point of F, S and T.

The uniqueness of the common fixed point z follows easily from (9). This completes the proof.

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