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NOTE ON VANDERMONDE'S CONVOLUTION THEOREM

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ABSTRACT The aim of this note is to prove Vandermonde's convolution theorem by using the theory of hypergeometric series as suggested in literature which does not seem to be easy to justify it We also provide an interesting identity and its application.

The generalized binomial coefficient $\binom{\alpha}{n}$ is defined by

$$\binom{\alpha}{n} := \begin{cases} \frac{\alpha(\alpha-1)\ldots(\alpha-n+1)}{n!} & \text{if } n = 1, 2, 3, \ldots ; \\ 1 & \text{if } n = 0, \end{cases} \tag{1}$$

α being any complex number.

Vandermonde [4] observed the following interesting identity involving the generalized binomial coefficient: For λ and μ any complex numbers,

$$\sum_{k=0}^{n} \binom{\lambda}{k}\binom{\mu}{n-k} = \binom{\lambda+\mu}{n}, \qquad n \geq 0, \tag{2}$$

which is usually referred to as Vandermonde's convolution theorem

The purpose of this note is to give a proof of (2) by using Gauss summation theorem in the theory of hypergeometric series as suggested in Srivastava *et al.* [3] and to provide an interesting identity involved in the Pochhammer symbol

The Gauss hypergeometric (or hypergeometric) series is defined by

$$\begin{aligned} {}_2F_1(a,b;\, c;\, z) &= F(a,b;\, c;\, z) \\ &= \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n\, n!} z^n, \quad c \neq 0, -1, -2, \ldots , \end{aligned} \tag{3}$$

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which was given by Carl Friedrich Gauss (1777-1855) [1] who in the year 1812 introduced this series into analysis and gave the F-notation for it, and the Pochhammer symbol (or the shifted factorial) $(\alpha)_n$ is defined by, α any complex number,

$$(4)\qquad (\alpha)_n := \begin{cases} \alpha(\alpha+1)\ldots(\alpha+n-1), & \text{if } n = 1, 2, 3, \ldots, \\ 1 & \text{if } n = 0. \end{cases}$$

Using the fundamental property $\Gamma(z+1) = z\Gamma(z)$ of the Gamma function Γ, $(\alpha)_n$ can be written in the form

$$(5)\qquad (\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)},$$

where Γ is the well-known Gamma function whose Weierstrass canonical product form is given by

$$(6)\qquad \{\Gamma(z)\}^{-1} = ze^{\gamma z} \prod_{k=1}^{\infty} \left(1+\frac{z}{k}\right) e^{-\frac{z}{k}},$$

γ being the Euler-Mascheroni's constant defined by

$$(7)\qquad \gamma = \lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \log n \right) \cong 0.577\,215\,664\,901\,532\ldots.$$

From definitions (1) and (4), we can easily deduce the following formulas:

$$(8)\qquad \binom{\alpha}{n} = \frac{(-1)^n(-\alpha)_n}{n!};$$

$$(9)\qquad (\alpha)_{n-k} = \frac{(-1)^k(\alpha)_n}{(1-\alpha-n)_k}.$$

Setting $\alpha = 1$ in (9) gives the natural property

$$(10)\qquad (-n)_k = \begin{cases} \frac{(-1)^k n!}{(n-k)!} & \text{if } 0 \le k \le n, \\ 0 & \text{if } k > n. \end{cases}$$

Gauss obtained the following important summation formula

$$(11) \qquad {}_2F_1(a,b,\,c;\,1) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n\, n!} = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)},$$

where $\mathrm{Re}(c-a-b) > 0$ ($c \neq 0, -1, -2, \ldots$).

An obvious special case of (11) occurs when the numerator parameter a or b is a nonpositive integer $-n$, say. We thus have a summation formula

$$(12) \qquad {}_2F_1(-n,b;\,c,\,1) = \frac{(c-b)_n}{(c)_n} \quad (n = 0, 1, 2, \ldots;\, c \neq 0, -1, -2, \ldots),$$

which incidentally can be shown to be equivalent to Vandermonde's convolution theorem (2)

Indeed, considering (10) and (12), we readily obtain

$$(c-b)_n = \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k (c)_n (b)_k}{(c)_k},$$

in which observing the relations

$$\frac{(c)_n}{(c)_k} = (-1)^{n-k}(-c-n+1)_{n-k}$$

and

$$(c-b)_n = (-1)^n (b-c-n+1)_n$$

yields

$$(b-c-n+1)_n = \sum_{k=0}^{n} \binom{n}{k} (b)_k (-c-n+1)_{n-k}.$$

Replacing b and $-c-n+1$ by λ and μ respectively in the formula just obtained, we have an interesting identity

$$(13) \qquad (\lambda+\mu)_n = \sum_{k=0}^{n} \binom{n}{k} (\lambda)_k (\mu)_{n-k},$$

in which we see a remarkable and surprising similarity to the binomial theorem

$$(\lambda+\mu)^n=\sum_{k=0}^{n}\binom{n}{k}\lambda^k\mu^{n-k}.$$

Now it is easy to see that (13) and (2) are equivalent by just applying the formula (8) to (2) and observing the resulting identity

$$(-\lambda-\mu)_n=\sum_{k=0}^{n}\binom{n}{k}(-\lambda)_k(-\mu)_{n-k}.$$

Next we will get an interesting identity involving the shifted factorial function. In fact, we would like to prove the following formula: For n any nonnegative integer, we have

$$\frac{(A)_n(B)_n}{(C)_n}=\sum_{k=0}^{n}\binom{n}{k}\frac{(C-B)_k(C-A)_k}{(C)_k}\cdot(A+B-C)_{n-k}, \tag{14}$$

where A, B, and C are any complex numbers with $C\neq 0,\,-1,\,-2,\,\ldots$.

We use induction on n to prove (14). It is easy to check that (14) is true for $n=0$ and $n=1$.

Assume that (14) is true for some nonnegative integer n. To complete the proof of the mathematical induction method, it is sufficient to show that (14) is true for $n+1$, i.e.,

$$\begin{aligned}&\frac{(A)_{n+1}(B)_{n+1}}{(C)_{n+1}}\\&=\sum_{k=0}^{n+1}\binom{n+1}{k}\frac{(C-B)_k(C-A)_k}{(C)_k}\cdot(A+B-C)_{n+1-k}.\end{aligned} \tag{15}$$

Letting the right-hand side of (15) by I, and considering an elementary identity related to the binomial coefficient

$$\binom{n+1}{k}=\binom{n}{k}+\binom{n}{k-1},$$

we obtain

$$I = \frac{(C-A)_{n+1}(C-B)_{n+1}}{(C)_{n+1}} + J, \tag{16}$$

where, using $(\alpha)_{n+1-k} = (\alpha+n-k)\,(\alpha)_{n-k}$,

$$\begin{aligned} J = (A+B-C+n)&\sum_{k=0}^{n}\binom{n}{k}\frac{(C-B)_k(C-A)_k}{(C)_k}\cdot(A+B-C)_{n-k} \\ &-\sum_{k=1}^{n}k\binom{n}{k}\frac{(C-B)_k(C-A)_k}{(C)_k}\cdot(A+B-C)_{n-k} \\ &+\sum_{k=1}^{n}\binom{n}{k-1}\frac{(C-B)_k(C-A)_k}{(C)_k}\cdot(A+B-C)_{n+1-k}. \end{aligned}$$

By the induction hypothesis, we readily have

$$\begin{aligned} J &= (A+B-C+n)\cdot\frac{(A)_n(B)_n}{(C)_n} \\ &\quad+\sum_{k=1}^{n}\frac{(C-B)_k(C-A)_k}{(C)_k}\cdot(A+B-C)_{n-k} \\ &\quad\times\left\{(A+B-C+n-k)\binom{n}{k-1}-k\binom{n}{k}\right\} \\ &= (A+B-C+n)\cdot\frac{(A)_n(B)_n}{(C)_n} \\ &\quad+\sum_{k=1}^{n}\binom{n}{k-1}\frac{(C-B)_k(C-A)_k}{(C)_k}\cdot(A+B-C-1)_{n+1-k}. \end{aligned}$$

Letting $k-1=k'$ in the last summation part and then dropping the prime on k, and arranging the resulting equation for the use of induction hypothesis, we readily obtain

$$\begin{aligned} J = (A+B-C+n)\cdot\frac{(A)_n(B)_n}{(C)_n} &- \frac{(C-A)_{n+1}(C-B)_{n+1}}{(C)_{n+1}} \\ + \frac{(C-A)(C-B)(A)_n(B)_n}{(C)_{n+1}}&. \end{aligned}$$

Finally setting the J into (16) immediately leads to

$$I = \frac{(A)_{n+1}(B)_{n+1}}{(C)_{n+1}},$$

which is just the desired result of the left-hand side of (15).

Putting $C = 2$ and $B = 1$ in (14), we obtain an interesting identity, which is a similar type of (13) and in reality a different one from (13),

$$(17) \qquad (\lambda + 2\mu)_n = \sum_{k=0}^{n} \binom{n+1}{k+1} (\lambda)_k (\mu)_{n-k},$$

λ and μ being any complex numbers with $\lambda + \mu = 1$.

For an application of (14), we first recall the generalized binomial theorem, α being any complex number,

$$(18) \qquad (1-z)^{-\alpha} = \sum_{n=0}^{\infty} \frac{(\alpha)_n}{n!} z^n, \qquad |z| < 1.$$

For any complex number z with $|z| < 1$, using (18) and (14),

$$\begin{aligned}
&(1-z)^{C-A-B} {}_2F_1(C-B, C-A;\, C;\, z) \\
&= \left\{ \sum_{n=0}^{\infty} \frac{(A+B-C)_n \, z^n}{n!} \right\} \left\{ \sum_{n=0}^{\infty} \frac{(C-B)_n (C-A)_n \, z^n}{(C)_n \, n!} \right\} \\
&= \sum_{n=0}^{\infty} \left\{ \sum_{k=0}^{n} \binom{n}{k} \frac{(C-B)_k (C-A)_k}{(C)_k} \cdot (A+B-C)_{n-k} \right\} \frac{z^n}{n!} \\
&= \sum_{n=0}^{\infty} \frac{(A)_n (B)_n}{(C)_n} \cdot \frac{z^n}{n!} \\
&= {}_2F_1(A, B;\, C;\, z).
\end{aligned}$$

We thus have, for any complex value z with $|z| < 1$,

$$(19) \qquad {}_2F_1(A, B;\, C;\, z) = (1-z)^{C-A-B} {}_2F_1(C-B, C-A;\, C;\, z),$$

where A, B, and C are any complex numbers with $C \neq 0, -1, -2, \ldots$. Note that (19) is a known result due to Euler (see Rainville [2, p. 60]).

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