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CAUCHY-SEQUENTIAL CONVERGENCE SPACES

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1. Introduction

It is well-known that every topological space can be specified completely by their convergent filters, but may not be done by their convergent sequences. Many topologists have characterized the class of topological spaces which can be determined by the knowledge of their convergent sequences([7]–[10]). A more penetrating solution was given by A. V. Arhangel'skii who called the spaces satisfying the following property(this property is called the *Fréchet-Urysohn property*([4] and [11])) *Fréchet spaces*([1]–[3]): The closure of any subset A of a topological space X is the set of all limits of sequences in A. Indeed, every first-countable space and so every metric space is Fréchet. Several authors introduced other generalizations of first-countable spaces and studied some properties of these spaces and their related topics([2]–[6], [11] and [12]). Recently, the author in [6] introduced sequential convergence structures and sequential convergence spaces and showed that Fréchet spaces are determined by these structures.

In this paper, we introduce the concept of Cauchy structures on a sequential convergence space and obtain a completion of a class of Cauchy-sequential convergence spaces.

2. Cauchy-sequential convergence spaces

In this section, we reintroduce sequential convergence structures and

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some properties of these structures which were announced in [6]. And we introduce the concepts of Cauchy-sequential convergence spaces, continuity in sequential convergence spaces and Cauchy-continuity in Cauchy-sequential convergence spaces. We give some examples.

Let X be a nonempty set and $S(X)$ the set of all sequences in X. Let N denote the set of all natural numbers. A nonempty subfamily L of the cartesian product $S(X) \times X$ is called *a sequential convergence structure on* X[6] if it satisfies the following properties:

(SC1) For each $x \in X$, $((x), x) \in L$, where (x) is the constant sequence whose k-th term is x for all indices $k \in N$.

(SC2) If $(\alpha, x) \in L$, then $(\beta, x) \in L$ for each subsequence β of α.

(SC3) Let $x \in X$ and $A \subset X$. If $(\alpha, x) \notin L$ for each $\alpha \in S(A)$, then $(\beta, x) \notin L$ for each $\beta \in S(\{y \in X | (\gamma, y) \in L$ for some $\gamma \in S(A)\})$.

If a sequential convergence structure L on X is given, then the pair (X, L) is called *a sequential convergence space*[6]. Hereafter, we use the notation $SC[X]$ for the set of all sequential convergence structures on X.

Let $(X, \mathcal{T})$ be a Fréchet space(we call $\mathcal{T}$ a Fréchet topology on X) and let $L_{\mathcal{T}}$ denote the set of all pairs $(\alpha, x) \in S(X) \times X$ such that the sequence α converges to x in the space $(X, \mathcal{T})$. Then, it is easy to check that $L_{\mathcal{T}} \in SC[X]$, and two topological spaces $(X, \mathcal{T})$ and $(X, L_{\mathcal{T}})$ are the very same since $(X, \mathcal{T})$ is a Fréchet space. Hence we have that every Fréchet space is a sequential convergence space. And, for each $L \in SC[X]$, define a function C_L of the power set $P(X)$ of X into itself as follows: $C_L(A) = \{x \in X | (\alpha, x) \in L$ for some $\alpha \in S(A)\}$ for all $A \in P(X)$. Then, C_L is a topological(or Kuratowski) closure operator on X and (X, C_L) is a Fréchet space. Let $\mathcal{L}(C_L)$ denote the set of all pairs $(\alpha, x) \in S(X) \times X$ such that α converges to x in the space (X, C_L). By the following example, we see that $L \subsetneqq \mathcal{L}(C_L)$, in general. Consequently, we have that every sequential convergence space (X, L) need not be a Fréchet space even if (X, L) determines a Fréchet space (X, C_L) as above.

EXAMPLE 2.1[6]. In general, $L \neq \mathcal{L}(C_L)$. Let Q be the set of all rational numbers with the usual topology and let $L_Q = \{(\alpha, x) \in S(Q) \times Q | \alpha$ converges to x in $Q\}$ and $L = \{((x), x) | x \in Q\} \cup \{(\alpha, x) \in S(Q) \times Q | \alpha$ converges to x in Q and α is either strictly increasing or strictly decreasing}. Then L, $L_Q \in SC[Q]$ and $L \subsetneq L_Q = \mathcal{L}(C_{L_Q}) = \mathcal{L}(C_L)$.

THEOREM 2.2[6]. *There exists a one-to-one correspondence* $\mathcal{T} \mapsto C_{L_\mathcal{T}}$ *between the set of all Fréchet topologies on a set* X *and* $\{C_L | L \in SC[X]\}$.

Throughout this paper, we assume that all given $L \in SC[X]$ satisfies the following property: *If* $(\alpha, x) \in L$ *and* $(\alpha, y) \in L$, *then* $x = y$. Note that the property above is called *unique sequential limits*. It is well-known that 'Hausdorffness' implies 'unique sequential limits', but the converse is not true(See [3]).

Let $\alpha \in S(X)$. For each $k \in N$, the k-th term of α is denoted by $\alpha(k)$. For each $\alpha, \beta \in S(X)$, we will use the notation $\alpha \wedge \beta$ for the sequence such that $\alpha \wedge \beta(2k-1) = \alpha(k)$ and $\alpha \wedge \beta(2k) = \beta(k)$ for each $k \in N$. Our notation $\alpha \wedge \beta$ is quoted from [7].

DEFINITION 2.3. Let $L \in SC[X]$. A nonempty subcollection C of $S(X)$ is called *a Cauchy structure* on the sequential convergence space (X, L) if it satisfies the following properties.

(C1) For each $(\alpha, x) \in L, \alpha \in C$.
(C2) If $\alpha \in C$, then $\beta \in C$ for each subsequence β of α.
(C3) If $\alpha, \beta \in C$ with common subsequence, then $\alpha \wedge \beta \in C$.
(C4) If $\alpha \wedge (x) \in C$, then $(\alpha, x) \in L$.

If a Cauchy structure C on a sequential convergence space (X, L) is given, then the triple (X, L, C) is called *a Cauchy-sequential convergence space*.

EXAMPLE 2.4. In Example 2.1, let $C = \{(x) | x \in Q\} \cup \{\alpha \in S(Q) | \alpha$ is Cauchy in Q in the usual sense and α is either strictly increasing or strictly decreasing$\}$ and let C_Q be the set of all Cauchy sequences in Q. Then, (Q, L, C) and (Q, L_Q, C_Q) are Cauchy-sequential convergence spaces.

By the usual ways, we can define the completeness of a Cauchy-sequential convergence space as follows:

DEFINITION 2.5. A Cauchy-sequential convergence space (X, L, C) is *complete* if and only if for each $\alpha \in C$, there exists $x \in X$ such that $(\alpha, x) \in L$.

Let R be the set of all real numbers with the usual topology and let $L_R = \{(\alpha, x) \in S(R) \times R | \alpha$ converges to x in R $\}$ and $C_R = \{\alpha \in S(R) | \alpha$ is Cauchy in R $\}$. Then it is obvious that (R, L_R, C_R) is a complete Cauchy-sequential convergence space, but (Q, L, C) and (Q, L_Q, C_Q) are not complete.

THEOREM 2.6. *Let (X, L, C) be a complete Cauchy-sequential convergence space and Y a nonempty subset of X. Then $(C_L(Y), L \cap (S(C_L(Y)) \times C_L(Y)), C \cap S(C_L(Y)))$ is a complete Cauchy-sequential convergence space.*

Proof. It is straightforward.

We next introduce the concepts of continuity and Cauchy-continuity.

DEFINITION 2.7. (1) A function $f : (X, L_X) \to (Y, L_Y)$ of a sequential convergence space (X, L_X) into a sequential convergence space (Y, L_Y) is called *continuous* if for each $(\alpha, x) \in L_X, (f(\alpha), f(x)) \in L_Y$ where $f(\alpha)$ denotes the image sequence of α under f.

(2) A function f of a Cauchy-sequential convergence space (X, L_X, C_X) into a Cauchy-sequential convergence space (Y, L_Y, C_Y) is called *Cauchy-continuous* if for each $\alpha \in C_X, f(\alpha) \in C_Y$.

The family of all real-valued continuous(Cauchy-continuous) functions defined on a sequential convergence space (X, L)(a Cauchy-sequential convergence space (X, L, C)) is denoted by $\mathbb{C}(X, L)$(resp. $\mathbb{C}(X, L, C)$). That is, $\mathbb{C}(X, L) = \{f | f : (X, L) \to (R, L_R)$ is continuous$\}$ and $\mathbb{C}(X, L, C) = \{f | f : (X, L, C) \to (R, L_R, C_R)$ is Cauchy-continuous$\}$.

REMARK. (1) It is well-known that $\mathbb{C}(Q, L_Q) = \mathbb{C}(Q, L_Q, C_Q)$.

(2) It is easy to show that if a Cauchy-sequential convergence space (X, L, C) is complete, then $\mathbb{C}(X, L) \subset \mathbb{C}(X, L, C)$. But, we have no any guarantee that if $f(\alpha) \in C_R$ for each $(\alpha, x) \in L$, then $f(x) = z$ where z is the limit of $f(\alpha)$ in R, i.e., $(f(\alpha), z) \in L_R$. Thus we have that in general, $\mathbb{C}(X, L) \neq \mathbb{C}(X, L, C)$ even if (X, L, C) is complete.

3. A completion of Cauchy-sequential convergence spaces

In this section, we investigate the completeness of real-valued function spaces defined on a Cauchy-sequential convergence space and obtain a completion of a class of Cauchy-sequential convergence spaces based on a real-valued function space.

Let X be a nonempty set and $\mathbb{L} \subset S(X) \times X$(need not be $\mathbb{L} \in SC[X]$). From now on, we need the following property:

($\star$) *Let $(\alpha, x) \in \mathbb{L}$ and let $((x_{nm}), \alpha(n)) \in \mathbb{L}$ for each $n \in N$. It is possible to choose a cross-sequence $(x_{nm(n)})$ in the double sequence (x_{nm}) such that $((x_{nm(n)}), x) \in \mathbb{L}$ and $m(n) \geqq n$ for all $n \in N$.*

Note that ($\star$) implies (SC3), but the converse is not true, in general. In Example 2.1, $L \in SC[Q]$ and so L satisfies (SC3). It is easy to check that this L does not satisfy ($\star$).

Let (X, L, C) be a Cauchy-sequential convergence space and let us adopt the following notations: $C^* = \{(f_n) \in S(\mathbb{C}(X, L, C)) |$ for each $\alpha \in C$, the double limit $\lim_{n,k} f_n(\alpha(k))$ exists$\}$ and $L^* = \{((f_n), f) | (f_n) \in C^*$ and $(f_n) \wedge (f) \in C^*$ for some $f \in \mathbb{C}(X, L, C)\}$

THEOREM 3.1. *Let (X, L, C) be a Cauchy-sequential convergence space. If L^* satisfies ($\star$), then $(\mathbb{C}(X, L, C), L^*, C^*)$ is a complete Cauchy sequential convergence space.*

Proof. Since L^* satisfies ($\star$), it is obvious that $L^* \in SC[\mathbb{C}(X, L, C)]$. We next prove that C^* is a Cauchy structure on $(\mathbb{C}(X, L, C), L^*)$. Clearly, C^* satisfies (C1) and (C2). It remains to prove that C^* satisfies (C3) and (C4).

(C3) Let $(f_n), (g_n) \in C^*$ with common subsequence. Assume that $f_{s(n)} = g_{t(n)}$, for all $n \in N$, where s and t denote strictly increasing functions of N into itself. Then for each $\alpha \in C$, $\lim_{n,k} f_n(\alpha(k)) = \lim_{n,k} f_{s(n)}(\alpha(k)) = \lim_{n,k} g_{t(n)}(\alpha(k)) = \lim_{n,k} g_n(\alpha(k))$, let r_α denote the limit. For each $\epsilon > 0$, there exist $n_1, n_2 \in N$ such that $|f_n(\alpha(k)) - r_\alpha| < \epsilon$ for each $n, k \geqq n_1$ and $|g_n(\alpha(k)) - r_\alpha| < \epsilon$ for each $n, k \geqq n_2$. Put $n_0 = max\{n_1, n_2\}$. It follows that $|(f_n) \wedge (g_n)(m)(\alpha(k)) - r_\alpha| < \epsilon$ for each $m, k \geqq 2n_0$. Hence $(f_n) \wedge (g_n) \in C^*$.

(C4) If $(f_n)\wedge(f) \in C^*$, then by definition of C^* we have $\lim_{m,k}(f_n)\wedge(f)(m)(\alpha(k)) = \lim_{n,k} f_n(\alpha(k)) = \lim_k f(\alpha(k))$ for each $\alpha \in C$, and the limit exists. Hence $(f_n) \in C^*$ by definition of C^*, and so $((f_n), f) \in L^*$ by definition of L^*.

Therefore, $(\mathbb{C}(X, L, C), L^*, C^*)$ is a Cauchy-sequential convergence space.

Finally, we show that the space $(\mathbb{C}(X, L, C), L^*, C^*)$ is complete. Let $(f_n) \in C^*$. Define a real-valued function f on X by $f(x) = \lim_n f_n(x)$ for all $x \in X$. Since $(x) \in C$, it is obvious that $\lim_n f_n(x)$ exists for each $x \in X$ and hence f is well-defined and $f \in \mathbb{C}(X, L, C)$. We now claim that $((f_n), f) \in L^*$. Let $\alpha \in C$. Since $(f_n) \in C^*$, $\lim_{n,k} f_n(\alpha(k))$ exists, and hence we have $\lim_{n,k} f_n(\alpha(k)) = \lim_k(\lim_n f_n(\alpha(k))) = \lim_n(\lim_k f_n(\alpha(k)))$. It follows that by definition of f, $\lim_{n,k} f_n(\alpha(k)) = \lim_k(\lim_n f_n(\alpha(k))) = \lim_k f(\alpha(k))$ and thus $((f_n), f) \in L^*$.

The proof completes.

REMARK. Let $C(L^*)$ denote the set of all sequences $\eta \in S(X)$ such that for each $((f_k), f) \in L^*$, $\lim_{n,k} f_k(\eta(n))$ exists. Then it is easy to check that $C \subset C(L^*)$, but $\lim_{n,k} f_k(\eta(n))$ need not be equal to $\lim_n f(\eta(n))$, i.e., $C \neq C(L^*)$.

Let $\mathbb{C}(\mathbb{C}(X, L, C), L^*, C^*)$ denote the set of all real-valued Cauchy-continuous functions defined on $(\mathbb{C}(X, L, C), L^*, C^*)$ and let us adopt the following two notations: $C^{**} = \{(\phi_n) \in S(\mathbb{C}(\mathbb{C}(X, L, C), L^*, C^*)) \mid lim_{n,k}\phi_n(f_k)$ exists for each $(f_k) \in C^*\}$ and $L^{**} = \{((\phi_n), \phi) \mid (\phi_n) \in C^{**}$ and $(\phi_n) \wedge (\phi) \in C^{**}$ for some $\phi \in \mathbb{C}(\mathbb{C}(X, L, C), L^*, C^*)\}$. Note that since $(\mathbb{C}(X, L, C), L^*, C^*)$ is complete by Theorem 3.1, we have that for each $(f_k) \in C^*$, $((f_k), f) \in L^*$ for some $f \in \mathbb{C}(X, L, C)$. Thus, $C^{**} = \{(\phi_n) \in S(\mathbb{C}(\mathbb{C}(X, L, C), L^*, C^*)) \mid \lim_{n,k} \phi_n(f_k) = \lim_n \phi_n(f)$ and the limit exists, for each $((f_k), f) \in L^*\}$.

The following theorem may be proved in much the same way as Theorem 3.1 and hence we omit the proof.

THEOREM 3.2. *Assume that the hypotheses of Theorem 3.1 are fulfilled and L^{**} satisfies $(\star)$. Then $(\mathbb{C}(\mathbb{C}(X, L, C), L^*, C^*),\ L^{**},\ C^{**})$ is a complete Cauchy-sequential convergence space.*

We require a definition. A function f of a Cauchy-sequential convergence space (X, L_X, C_X) into a Cauchy-sequential convergence space

(Y, L_Y, C_Y) is called *a Cauchy-embedding* if f is injective and f and f^{-1} are both Cauchy-continuous.

THEOREM 3.3. *Assume that the hypotheses of Theorems 3.1-3.2 are fulfilled and $\mathbb{C}(X, L, C)$ separates points of X. Define a function $i : (X, L, C) \to (\mathbb{C}(\mathbb{C}(X, L, C), L^*, C^*),\ L^{**},\ C^{**})$ by $i(x)(f) = f(x)$ for each $x \in X$ and $f \in \mathbb{C}(X, L, C)$. Then i is a Cauchy-embedding if and only if $C = C(L^*)$.*

Proof. Injectiveness of i: Let x and y be any distinct points of X. Then, by the assumption; $\mathbb{C}(X, L, C)$ separates points, there exists $f \in \mathbb{C}(X, L, C)$ such that $f(x) \neq f(y)$. It follows that $i(x)(f) = f(x) \neq f(y) = i(y)(f)$ and hence $i(x) \neq i(y)$.

Cauchy-continuity of i: Let $\alpha \in C$ and $((f_k), f) \in L^*$. Then, by definitions of L^* and C^*, it is clear that $(f_k) \in C^*$ and $\lim_{n,k} f_k(\alpha(n))$ exists. Since $((f_k), f) \in L^*$, we have $\lim_{n,k} f_k(\alpha(n)) = \lim_n f(\alpha(n))$. It follows that $\lim_{n,k} i(\alpha(n))(f_k) = \lim_{n,k} f_k(\alpha(n)) = \lim_n f(\alpha(n))$ and the limit exists. By Theorem 3.1, $(\mathbb{C}(X, L, C), L^*, C^*)$ is complete and thus $i(\alpha) \in C^{**}$.

Cauchy-continuity of i^{-1}: Let $(\phi_n) \in C^{**}$ with the range of (ϕ_n) is contained in $i(X)$ and let $((f_k), f) \in L^*$. Then, by definition of C^{**}, $\lim_{n,k} \phi_n(f_k) = \lim_n \phi_n(f)$ and the limit exists. In fact, since $(\mathbb{C}(\mathbb{C}(X, L, C), L^*, C^*),\ L^{**},\ C^{**})$ is complete by Theorem 3.2, $((\phi_n), \phi) \in L^{**}$ for some $\phi \in \mathbb{C}(\mathbb{C}(X, L, C), L^*, C^*)$ and $\lim_n \phi_n(f) = \phi(f)$. Since $\lim_{n,k} f_k(i^{-1}(\phi_n) = \lim_{n,k} i(i^{-1}(\phi_n))(f_k) = \lim_{n,k} \phi_n(f_k)$, we have $(i^{-1}(\phi_n)) \in C(L^*)$ by definition of $C(L^*)$ and hence, by hypothesis; $C = C(L^*)$, $(i^{-1}(\phi_n)) \in C$.

Therefore, i is a Cauchy-embedding.

Conversely, assume that i is a Cauchy-embedding and let $\eta \in C(L^*)$. By definition of $C(L^*)$, for each $((f_k), f) \in L^*$, $\lim_{n,k} f_k(\eta(n)) = \lim_n f(\eta(n))$ and the limit exists. Since $\lim_{n,k} f_k(\eta(n)) = \lim_{n,k} i(\eta(n))$ (f_k), we have $i(\eta) \in C^{**}$ by definition of C^{**}. By injectiveness and Cauchy-continuity of i^{-1}, we have $\eta \in C$. Therefore, $C(L^*) \subset C$. As $C \subset C(L^*)$ is always true, it follows that $C = C(L^*)$.

The proof completes.

Let (X, L_X, C_X) be a non-complete Cauchy-sequential convergence space. A complete Cauchy-sequential convergence space (Y, L_Y, C_Y) is called *a completion of* (X, L_X, C_X) if there exists a Cauchy-embedding $f : (X, L_X, C_X) \to (Y, L_Y, C_Y)$ such that $C_{L_Y}(f(X)) = Y$. This involves no loss of generality.

According to Theorem 2.6 and Theorems 3.1–3.3, we consequently obtain the following result.

THEOREM 3.4. *Let* (X, L, C) *be a non-complete Cauchy-sequential convergence space. Assume that the hypotheses of Theorems 3.1–3.3 are fulfilled and* $C = C(L^*)$. *Then,* $(C_{L^{**}}(i(X)),\ L^{**} \cap (S(C_{L^{**}}(i(X))) \times C_{L^{**}}(i(X))),\ C^{**} \cap S(C_{L^{**}}(i(X))))$ *is a completion of* (X, L, C).

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