East Asian Math J. **14** (1998), No. 1, pp. 135–140

A STUDY ON D.G. NEAR-RINGS AND THEIR MODULES

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1. Introduction

A near-ring is a nonempty set R with two binary operations $+$ and such that $(R,+)$ is a group(not necessarily abelian) with identity 0, $(R,\cdot)$ is a semigroup and $a(b+c) = ab + ac$ for all a, b, c in R. In general a near-ring R with the extra axiom $0a = 0$ for all $a \in R$ is said to be zero symmetric. An element d in R is called distributive if $(a+b)d = ad + bd$ for all a and b in R. Let $(G,+)$ be a group(not necessarily abelian). If we set $M(G) := \{f \mid f : G \longrightarrow G\}$, and define the sum $f + g$ of any two mappings f, g in $M(G)$ by the rule $x(f+g) = xf + xg$ for all $X \in G$ and the product $f \cdot g$ by the rule $x(f \cdot g) = (xf)g$ for all $x \in G$ then $(M(G), +, \cdot)$ forms a near-ring. Let $M_0(G) := f \in M(G) \mid 0f = 0$. Then $(M_0(G), +, \cdot)$ is a zero symmetric near-ring. For the remainder results and definitions on near-rings, we refer to G. Pilz [6].

Let R be any near-ring and G an additive group. Then G is called an $R-$group (or module) if there exists a near-ring homomorphism

$$\theta : (R, +, \cdot) \longrightarrow (M(G), +, \cdot).$$

Such a homomorphism θ is called a representation of R on G, we will write that xr for $x(r\theta)$ for all $x \in G$ and $r \in R$. A representation θ is called faithful if $Ker\theta = 0$.

The near-ring R is called a distributively generated(briefly, D.G.) near-ring if $(R,+) = gp < S >$ where S is a semigroup of distributive elements in R, we denote it (R, S). The distributive elements of $M_0(G)$

Received March 11, 1998.

This paper was supported (in part) by NON DIRECTED RESEARCH FUND, Mooryang Hyang Research Foundation

are $End(G)$, the semigroup of all the endomorphisns of the group G. Here we denote that $E(G)$ is the D.G. near-ring generated by $End(G)$, and call that $E(G)$ is the endomorphism near-ring of the group G.A homomorphism

$$\theta : (R, S) \longrightarrow (T, U)$$

is a D.G. near-ring homomorphism if θ is a near-ring homomorphism such that $S\theta \subseteq U$. A semigroup homomorphism $\theta : S \longrightarrow U$ is a D.G. near-ring homomorphism if it is a group homomorphism from $(R, +)$ to $(T, +)$. See C. G. Lyons and J. D. P. Meldrum([3],[4]).

Let R be a near-ring and let G be an $R-$group. If there exists x in G such that $G = xR$, that is, $G = \{xr \mid r \in R\}$, then G is called a monogenic $R-$group and the element x is called a generator of G. See J. D. P. Meldrum and G. Pilz([5], [6]).

2. Properties of D.G. near-rings (R,S) and D.G. $(R,S)-$modules

Now we may introduce new concepts as follows: Let (R, S) be a D.G. near-ring. Then an additive group G is called a D.G. $(R, S)-$group(or D.G. $(R, S)-$module) if there is a near-ring homomorphism

$$\theta : (R, S) \longrightarrow (E(G), End(G))$$

such that $S\theta \subseteq End(G)$. Such a homomorphism is called a D.G. representation of (R, S). This D.G. representation is said to be faithful if $Ker\theta = 0$.

LEMMA 2.1[5]. *Let (R, S) be a D.G. near-ring. Then all $R-$subgroups and all $R-$homomorphic images of a D.G. $(R, S)-$group are D.G. $(R, S)-$groups.*

Next, let R be a near-ring and G an additive group. If there is a scalar multiplication

$$\theta : (R, S) \longrightarrow G$$

which is defined by $\theta(a, x) = ax$ such that $(ab)x = a(bx)$ and $a(x+y) = ax = ay$ for all $a, b \in R$ and for all $x, y \in G$, Then G is called a $R-$cogroup(or comodule), see Y. U. Cho[2]. If R is a right near-ring, then every $R-$cogroup is an $R-$group for R as an $R-$group. Similar method of lemma 2.1 shows the following lemma:

LEMMA 2.2. *Let (R,S) be a D.G. near-ring. Then all $R-$subgroups and all $R-$homomorphic images of a D.G. $(R,S)-$cogroup are D.G. $(R,S)-$cogroups.*

PROPOSITION 2.3. *Let (R,S) be a D.G. near-ring. Then*

(1) *Every monogenic $R-$group is a D.G. $(R,S)-$group.*

(2) *Every monogenic $R-$cogroup is a D.G. $(R,S)-$cogroup.*

Proof. Let G be a monogenic $R-$group with x as a generator. Then the map $\phi : r| \longrightarrow xr$ is an $R-$epimorphism from R to G as $R-$groups. We see that

$$G \cong R/A(x),$$

where $A(x) = (0 : x) = Ker\phi$. See for this notation Y. U. Cho[2]. From the Lemma 2.1, we obtain that G is a D.G. $(R,S)-$group.

For G is a monogenic $R-$cogroup with x as a generator, the map $\psi : r| \longrightarrow rx$ is also an $R-$epimorphism from R to G as an $R-$cogroups. Thus we have that

$$G \cong R/Ann(x),$$

where $Ann(x) = [0 : x] = Ker\psi$. See also for this notation Y. U. Cho[2]. By the Lemma 2.2, we see that G is a D.G. $(R,S)-$cogroup. $\square$

THEOREM 2.4. *Let (R,S) be a D.G. near-ring and $(G,+)$ is an abelian group. Then*

(1) *If G is a faithful D.G. $(R,S)-$group, then R is a ring.*

(2) *If G is a faithful D.G. $(R,S)-$cogroup, then R is also a ring.*

Proof. (1) Let $x \in G$ and $r,s \in R$. Then, since $(G,+)$ is abelian,

$$x(r+s) = xr + xs = xs + xr = x(s+r).$$

Thus we get that $x(r+s) - (s+r) = 0$ for all $x \in G$, that is, $(r+s) - (s+r) \in Ker\theta = (0 : G) = A(x)$, where $\theta : R \longrightarrow M(G)$ is a representation of R on G. Since G is faithful, that is, θ is faithful,

$Ker\theta = (0 : G) = 0$. Hence for all $r, s \in R, r + s = s + r$. Consequently, $(R, +)$ is abelian.

Next we must show that R satisfies the right distributive law. Obviously, we note that for all $r, r' \in R$ and all $s \in S$,

$$0s = 0,\ (-r)s = -(rs) = r(-s) \text{ and } (r + r')s = rs + r's.$$

Let $x \in G$ and $r, s, t \in R$. Then the element t in R is represented by

$$t = \delta_1 s_1 + \delta_2 s_2 + \delta_3 s_3 + \cdots + \delta_n s_n,$$

where $\delta_i = 1$, or -1 and $s_i \in S$ for $1 \leq i \leq n$. Thus, using the above note and $(G, +)$ is abelian, we have the following equalities:

$$\begin{aligned}
x(r+s)t &= (xr + xs)t = (xr + xs)(\delta_1 s_1 + \delta_2 s_2 + \cdots + \delta_n s_n) \\
&= (xr + xs)\delta_1 s_1 + (xr + xs)\delta_2 s_2 + \cdots + (xr + xs)\delta_n s_n \\
&= \delta_1(xr + xs)s_1 + \delta_2(xr + xs)s_2 + \cdots + \delta_n(xr + xs)s_n \\
&= \delta_1(xrs_1 + xss_1) + \delta_2(xrs_2 + xss_2) + \cdots + \delta_n(xrs_n + xss_n) \\
&= \delta_1 xrs_1 + \delta_1 xss_1 + \delta_2 xrs_2 + \delta_2 xss_2 + \cdots + \delta_n xrs_n + \delta_n xss_n \\
&= xr\delta_1 s_1 + xs\delta_1 s_1 + xr\delta_2 s_2 + xs\delta_2 s_2 + \cdots + xr\delta_n s_n + xs\delta_n s_n \\
&= xr(\delta_1 s_1 + \delta_2 s_2 + \cdots + \delta_n s_n) + xs(\delta_1 s_1 + \delta_2 s_2 + \cdots \delta_n s_n) \\
&= xrt + xst = x(rt + st).
\end{aligned}$$

thus we obtain that $x(r + s)t - (rt + st) = 0$ for all $x \in G$, namely,

$$(r + s)t - (rt + st) \in (0 : G) = A(G).$$

Also using G is faithful, that is, $A(G) = 0$. Applying the beginning part of this proof, we see that $(r + s)t = rt + st$ for all $r, s, t \in R$, consequently, R satisfies the right distributivr law. Hence R becomes a ring.
(2) We can prove this as similar method to the proof of (1). □

As an immediate consequence of theorem 2.4, we have the following important corollary.

Corollary 2.5. *Let (R, S) be an abelian D.G. near-ring. Then R is a ring.*

Finally, we may define a new concept and then characterize D.G. near-ring with this new concept as following.

A near-ring R is called generalized right bipotent if for all $a \in R$ there exists a positive integer n such that

$$a^n R = a^{n+1} R.$$

There are many examples of generalized right bipotent near-rings, for example, Boolean near-rings.

Theorem 2.6. *Let (R, S) be a generalized right bipotent D.G. near-ring. If there exists an element in R which is not a zero divisor, then R has an identity.*

Proof. Let $a \in R$ such that a is not a zero divisor then also a^n is not a zero divisor for any positive integer n. Indeed, suppose that a^n is a zero divisor, then there exists a nonzero element $x \in R$ such that $a^n x = 0$, that is, $a(a^{n-1}x) = 0$, since a is not a zero divisor, this implies that $a^{n-1}x = 0$. Continuing this procedure we get that $x = 0$, this fact is a contradiction. Hence a^n is not a zero divisor.

Assume that $a \in R$ is not a zero divisor which is not zero. Since R is generalized right bipotent, we have the following equation

$$a^n R = a^{n+1} R$$

for some positive integer n. This implies that $a^n a = a^{n+1} e$ for some e in R, that is, $a^n(a - ae) = 0$. From the above remark of this proof, since a^n is not a left zero divisor, we obtain that $a = ae$. Also, from the equation $a(ea - a) = a(ea) - aa = (ae)a - aa = aa - aa = 0$, we get that $a = ea$.

Next, let r be an arbitrary element of R. From the following equation:

$$a(er - r) = a(er) - ar = (ae)r - ar = ar - ar = 0,$$

since a is not a left zero divisor, we obtain that $er = r$, so that e is the left identity of R.

Finally, let r be any element of R. Suppose a is not a zero divesor on R, Then since (R, S) is a D,G. near-ring, there exists a positive integer n, we can decompose a as follows:

$$a = \delta_1 s_1 + \delta_2 s_2 + \cdots + \delta_n s_n$$

, for some $s_i \in S, \delta_i = 1$ or -1 for $1 \leq i \leq n$. Then we have the following equalities:

$$\begin{aligned}(re - r)a &= (re - r)(\delta_1 s_1 + \delta_2 s_2 + \cdots + \delta_n s_n) \\ &= (re - r)\delta_1 s_1 + (re - r)\delta_2 s_2 + \cdots + (re - r)\delta_n s_n \\ &= \delta_1 (re - r)s_1 + \delta_2 (re - r)s_2 + \cdots + \delta_n (re - r)s_n \\ &= \delta_1 (res_1 - rs_1) + \delta_2 (res_2 - rs_2) + \cdots + \delta_n (res_n - rs_n) \\ &= \delta_1 (rs_1 - rs_1) + \delta_2 (rs_2 - rs_2) + \cdots + \delta_n (rs_n - rs_n) \\ &= 0 + 0 + \cdots + 0 = 0.\end{aligned}$$

This implies that $re = r$, that is, e is the right identity of R. Consequently, e is the identity of R. □

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