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CERTAIN CLASS OF FRACTIONAL CALCULUS OPERATOR WITH TWO FIXED POINTS

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ABSTRACT. This paper deals with functions of the form $f(z) = a_1 z - \sum_{n=2}^{\infty} a_n z^n$ $(a_1 > 0, a_n \geqslant 0)$ with $(1-\mu)f(z_0)/z_0 + \mu f'(z_0) = 1$ $(-1 < z_0 < 1)$ We introduce the class $\varphi(\mu, \eta, \gamma, \delta, A, B, z_0)$ with generalized fractional derivatives. Also we have obtained coefficient inequalities, distortion theorem and radious problem of functions belonging to the calss $\varphi(\mu, \eta, \gamma, \delta, A, B; z_0)$.

1. Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n \quad (a_n \geqslant 0) \tag{1.1}$$

which are univalent in unit disk $D = \{z : |z| < 1\}$. Recently, Uralegaddi and Somanatha [4] studied the class of functions of the form

$$f(z) = a_1 z - \sum_{n=2}^{\infty} a_n z^n \quad (a_1 > 0,\ a_n \geqslant 0) \tag{1.2}$$

with

$$\frac{(1-\mu)f(z_0)}{z_0} + \mu f'(z_0) = 1, \tag{1.3}$$

where

$$-1 < z_0 < 1, \quad 0 \leq \mu \leq 1.$$

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A function $f(z)$ is said to be convex of order α, if

$$Re\left\{1+\frac{zf''(z)}{f(z)}\right\}>\alpha \quad (z\in D:0\leq\alpha<1) \tag{1.4}$$

We denote by $C^*(\alpha)$ the class of convex functions of order α $(0\leq\alpha<1)$.

Let $f(z)$ be analytic function and $g(z)$be multivalent function satisfying $f(0)=g(0)$ and $f(D)\subset g(D)$, then $f(z)$ is said to be subordinate to $g(z)$, and is denoted by $f(z)\prec g(z)$.

We now recall the following definition of a generalized fractional operator introduced by Srivastava et al [3].

DEFINITION 1. For real numbers η $(\eta>0)$, γ, and δ, the generalized fractional integral operator $I_{0,z}^{\eta,\gamma,\delta}$ of order η is defined, for a function $f(z)$, by

$$\begin{aligned} &I_{0,z}^{\eta,\gamma,\delta}f(z)\\ &=\frac{z^{-\eta-\gamma}}{\Gamma(\eta)}\int_0^z(z-\xi)^{\eta-1}F\left(\eta+\gamma,-\delta;\eta;1-\frac{\xi}{z}\right)f(\xi)d\xi, \end{aligned} \tag{1.5}$$

where $f(z)$ is an analytic function in a simply-connected region of the z-plane containing the origin with the order

$$f(z)=O(|z|^{\varepsilon}),\ (z\longrightarrow 0),\ (\varepsilon>\max\{0,\gamma-\delta\}-1),$$

$$F(a,b;c;z)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!} \quad (z\in D), \tag{1.6}$$

and $(\nu)_n$ being the pochhammer symbol defined by

$$(\nu)_n=\frac{\Gamma(\nu+n)}{\Gamma(\nu)}=\begin{cases}1, & (n=0)\\ \nu(\nu+1)\cdots(\nu+n-1) & (n\in N=\{1,2,\cdots\}),\end{cases}$$

provided further that the multiplicity of $(z-\xi)^{\eta-1}$ is removed by requiring $\log(z-\xi)$ to be real when $(z-\xi)>0$.

DEFINITION 2. For real numbers $\eta(0 \leq \eta < 1)$, γ, and δ, the generalized fractional derivative operator $J_{0,z}^{\eta,\gamma,\delta}$ of order η is defined, for a function $f(z)$, by

(1.7)

$$J_{0,z}^{\eta,\gamma,\delta} f(z) = \frac{1}{\Gamma(1-\eta)} \frac{d}{dz} \left\{ z^{\eta-\gamma} \int_0^z (z-\xi)^{-\eta} F\left(\gamma-\eta, -\delta; 1-\eta; 1-\frac{\xi}{z}\right) f(\xi) d\xi \right\},$$

where $f(z)$ is an analytic function in a simply-connected region of the z-plane containing the origin, and the myltiplicity of $(z-\xi)^{-\eta}$ is removed as Definition 1 above.

LEMMA 1 [3]. *If $0 \leq \eta < 1$ and $k > \gamma - \delta - 2$, then*

$$J_{0,z}^{\eta,\gamma,\delta} z^k = \frac{\Gamma(k+1)\Gamma(k-\gamma+\delta+2)}{\Gamma(k-\gamma+1)\Gamma(k-\eta+\delta+2)} z^{k-\gamma}. \tag{1.8}$$

We will define the following definition.

DEFINITION 3. *A function $f(z)$ defined by (1.2) and satisfying (1.3) is said to be in the class $\varphi(\mu, \eta, \gamma, \delta, A, B; z_0)$ if*

$$\frac{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}{\Gamma(3-\gamma+\delta)} z^{\gamma-1} J_{0,z}^{\eta,\gamma,\delta} f(z) \prec a_1 \frac{1+Az}{1+Bz}$$

where $0 \leq \eta < 1$, $\gamma < 2$, $\eta - \delta < 3$, $\gamma - \delta < 3$, $-1 \leq B < A \leq 1$ and $-1 \leq B \leq 0$.

For $\eta = \gamma$, $\varphi(\mu, \eta, \eta, \delta, A, B; z_0)$ has been studied by S. R. Kulkarni and U. H. Naik [1]. The main purpose of this paper is to investigate coefficient inequalities, distortion theorem and radious problem of functions belonging to the calss $\varphi(\mu, \eta, \gamma, \delta, A, B; z_0)$

2. Coefficient inequalities

THEOREM 1. *A function $f(z)$ belongs to $\varphi(\mu,\eta,\gamma,\delta,A,B;z_0)$ if, and only if*

$$\sum_{n=2}^{\infty}\left\{\frac{(1-B)}{(A-B)}\phi(\eta,\gamma,\delta,n)-[(1-\mu)+n\mu]z_0^{n-1}\right\}a_n\le 1, \tag{2.1}$$

where

$$\phi(\eta,\gamma,\delta,n)=\frac{(3-\gamma+\delta)_{n-1}n!}{(2-\gamma)_{n-1}(3-\eta+\delta)_{n-1}}$$

and $(v)_n$ is Pochhammer symbol.

Proof. Suppose $f(z)$ belongs to $\varphi(\mu,\eta,\gamma,\delta,A,B;z_0)$. Then we have

$$F(z)=a_1\frac{1+Aw(z)}{1+Bw(z)}\quad(-1\le B<A\le 1) \tag{2.2}$$

where $w(z)$ is analytic in D with $w(0)=0$, $|w(z)|<1$ and

$$\begin{aligned}F(z)=&\frac{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}{\Gamma(3-\gamma+\delta)}z^{\gamma-1}J_{0,z}^{\eta,\gamma,\delta}f(z)\\=&a_1-\sum_{n=2}^{\infty}\phi(\eta,\gamma,\delta,n)a_nz^{n-1},\end{aligned}$$

and

$$\phi(\eta,\gamma,\delta,n)=\frac{(3-\gamma+\delta)_{n-1}n!}{(2-\gamma)_{n-1}(3-\eta+\delta)_{n-1}}.$$

Equation (2.2) is equivalent to

$$\left|\frac{F(z)-a_1}{BF(z)-a_1A}\right|=|w(z)|<1. \tag{2.3}$$

Since $|Re\ z|\le|z|$ for any z, we have from (2.3)

$$Re\left\{\frac{\sum_{n=2}^{\infty}\phi(\eta,\gamma,\delta,n)a_nz^{n-1}}{(A-B)a_1+\sum_{n=2}^{\infty}B\phi(\eta,\gamma,\delta,n)a_nz^{n-1}}\right\}<1. \tag{2.4}$$

Choose values of z on the real axis so that $F(z)$ is real, upon cleaning the denominator in (2.4) and letting $z \to 1$ through the real values, we get

$$\sum_{n=2}^{\infty}(1-B)\phi(\eta,\gamma,\delta,n)a_n \le a_1(A-B). \tag{2.5}$$

Finally substiuting $a_1 = 1 + \sum_{n=2}^{\infty}[(1-\mu)+n\mu]a_n z_0^{n-1}$ in (2.5), we get (2.1).

Conversely, suppose that (2.1) holds. Consider

$$\begin{aligned}
&|F(z)-a_1| - |BF(z)-a_1A| \\
&= \left|\sum_{n=2}^{\infty}\phi(\eta,\gamma,\delta,n)a_n z^{n-1}\right| - \left|(A-B)a_1 + \sum_{n=2}^{\infty}B\phi(\eta,\gamma,\delta,n)a_n z^{n-1}\right| \\
&\le \sum_{n=2}^{\infty}(1-B)\phi(\eta,\gamma,\delta,n)a_n - a_1(A-B) \\
&\le 0, \quad \text{by hypothesis.}
\end{aligned}$$

Hence, by maximum modulus theorem, we get

$$\left|\frac{F(z)-a_1}{BF(z)-a_1A}\right| < 1 \quad (z \in D),$$

which implies that there exist an analytic function $w(z)$ such that $w(0) = 0$ and $|w(z)| < 1$ and that

$$\frac{F(z)-a_1}{BF(z)-a_1A} = w(z)$$

which in turn implies that $f(z)$ belongs to $\varphi(\mu,\eta,\gamma,A,B;z_0)$.

3. A distortion theorem

THEOREM 2. *If a function $f(z)$ is in the class $\varphi(\mu, \eta, \gamma, A, B; z_0)$ with $3\eta \geq \gamma(\eta - \delta - 1)$, then*

$$\begin{aligned} & a_1\left(|z| - \frac{(A-B)(2-\gamma)(3-\eta+\delta)}{2(3-\gamma+\delta)(1-B)}|z|^2\right) \leq |f(z)| \\ & \leq a_1\left(|z| + \frac{(A-B)(2-\gamma)(3-\eta+\delta)}{2(3-\gamma+\delta)(1-B)}|z|^2\right) \quad (z \in D). \end{aligned} \tag{3.1}$$

Proof. In view of equation (2.5) and the fact that $\phi(\eta, \gamma, \delta, n)$ is non-decreasing for $n \geq 2$, we have

$$\begin{aligned} & \frac{2(3-\gamma+\delta)(1-B)}{(2-\gamma)(3-\eta+\delta)} \sum_{n=2}^{\infty} a_n \\ & \leq \sum_{n=2}^{\infty} \phi(\eta, \gamma, \delta, n)(1-B)a_n \leq a_1(A-B), \end{aligned} \tag{3.2}$$

which is equivalent to

$$\sum_{n=2}^{\infty} a_n \leq \frac{a_1(A-B)(2-\gamma)(3-\eta+\delta)}{2(3-\gamma+\delta)(1-B)}. \tag{3.3}$$

Therefore, we obtain

$$\begin{aligned} & |f(z)| \geq a_1|z| - |z|^2 \sum_{n=2}^{\infty} a_n \\ & \geq a_1\left(|z| - \frac{(A-B)(2-\gamma)(3-\eta+\delta)}{2(3-\gamma+\delta)(1-B)}|z|^2\right) \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} & |f(z)| \leq a_1|z| + |z|^2 \sum_{n=2}^{\infty} a_n \\ & \leq a_1\left(|z| + \frac{(A-B)(2-\gamma)(3-\eta+\delta)}{2(3-\gamma+\delta)(1-B)}|z|^2\right). \end{aligned} \tag{3.5}$$

THEOREM 3. *If a function $f(z)$ is in the class $\phi(\mu,\eta,\gamma,\delta,A,B;z_0)$, then*

$$(3.6)\quad |J_{0,z}^{\eta,\gamma,\delta}f(z)| \geq \frac{a_1\Gamma(3-\gamma+\delta)}{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}\left(|z|^{1-\gamma}-\frac{A-B}{1-B}|z|^{2-\gamma}\right)$$

and

$$(3.7)\quad |J_{0,z}^{\eta,\gamma,\delta}f(z)| \leq \frac{a_1\Gamma(3-\gamma+\delta)}{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}\left(|z|^{1-\gamma}+\frac{A-B}{1-B}|z|^{2-\gamma}\right)$$

for $z \in D_0$, where

$$D_0 = \begin{cases} D, & \gamma \leq 1 \\ D-\{0\}, & 1<\gamma<2. \end{cases}$$

Proof. By using second inequality in (3.1), we observe that

$$(3.8)\quad \begin{aligned} &\left|\frac{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}{\Gamma(3-\gamma+\delta)}z^{\gamma}J_{0,z}^{\eta,\gamma,\delta}f(z)\right| \\ &\geq a_1|z|-\sum_{n=2}^{\infty}\phi(\eta,\gamma,\delta,n)a_n|z|^n \\ &\geq a_1|z|-|z|^2\left(\sum_{n=2}^{\infty}\phi(\eta,\gamma,\delta,n)a_n\right) \\ &\geq a_1\left(|z|-\frac{A-B}{1-B}|z|^2\right), \end{aligned}$$

which is equivalent to (3.6).

Next

$$\begin{aligned} &\left|\frac{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}{\Gamma(3-\gamma+\delta)}z^{\gamma}J_{0,z}^{\eta,\gamma,\delta}f(z)\right| \\ &\leq a_1|z|+\sum_{n=2}^{\infty}\phi(\eta,\gamma,\delta,n)a_n|z|^n \\ &\leq a_1|z|+|z|^2\left(\sum_{n=2}^{\infty}\phi(\eta,\gamma,\delta,n)a_n\right) \\ &\leq a_1\left(|z|+\frac{A-B}{1-B}|z|^2\right), \end{aligned}$$

which yields (3.7).

COROLLARY 1. *Let a function $f(z)$ belong to $\varphi(\mu,\eta,\gamma,\delta,A,B;z_0)$ with $3\eta \geq \gamma(\eta-\delta-1)$. Then $f(z)$ is included in a disk with its center at origin and radious r given by*

$$r = a_1\left(1+\frac{(A-B)(2-\gamma)(3-\eta+\delta)}{2(3-\gamma+\delta)(1-B)}\right),$$

and $J_{0,z}^{\eta,\gamma,\delta}f(z)$ is included in a disk with its center at the origin and radious R given by

$$R = \frac{a_1\Gamma(3-\gamma+\delta)}{\Gamma(2-\gamma)\Gamma(3-\eta+\delta)}, \quad \left(1+\frac{A-B}{1-B}\right).$$

4. Radious of convexity

THEOREM 4. *Let $f(z)$ belong to $\varphi(\mu,\eta,\gamma,\delta,A,B;z_0)$. Then $f(z)$ is convex in the disk*

$$|z| < r = r(\eta,\gamma,\delta,A,B) = \inf_{\substack{n\geq 2\\ n\in\mathbf{N}}}\left(\frac{(1-B)\phi(\eta,\gamma,\delta,n)}{n^2(A-B)}\right)^{\frac{1}{n-1}}. \tag{4.1}$$

The result is sharp for the function given by

$$f(z) = \frac{(1-B)\phi(\eta,\gamma,\delta,n)z-(A-B)z^n}{\{(1-B)\phi(\eta,\gamma,\delta,n)-[(1-\mu)+n\mu](A-B)z_0^{n-1}\}}. \tag{4.2}$$

Proof. It is sufficient to prove that $\left|\frac{zf''(z)}{f'(z)}\right| \leq 1$ for $|z| < r(\eta,\gamma,\delta,A,B)$. A simple calculation gives us

$$\left|\frac{zf''(z)}{f'(z)}\right| \leq \frac{\sum\limits_{n=2}^{\infty} n(n-1)a_n|z|^{n-1}}{a_1-\sum\limits_{n=2}^{\infty} na_n|z|^{n-1}}.$$

Clearly, $\left|\frac{zf''(z)}{f'(z)}\right| \leq 1$ if

$$\sum_{n=2}^{\infty} n(n-1)a_n|z|^{n-1} \leq a_1 - \sum_{n=2}^{\infty} na_n|z|^{n-1}. \tag{4.3}$$

Using $a_1 = 1 + \sum\limits_{n=2}^{\infty}[(1-\mu)+n\mu]z_0^{n-1}$ in (4.3), we are led to

$$\sum_{n=2}^{\infty} a_n\{n^2|z|^{n-1} - [(1-\mu)+n\mu]z_0^{n-1}\} \leq 1. \tag{4.4}$$

By Theorem 1, we have

$$\sum_{n=2}^{\infty} a_n \left\{\frac{(1-B)}{(A-B)}\phi(\eta,\gamma,\delta,n) - [(1-\mu)+n\mu]z_0^{n-1}\right\} \leq 1.$$

Hence (4.4) will hold, if

$$n^2|z|^{n-1} - [(1-\mu)+n\mu]z_0^{n-1} \leq \frac{(1-B)}{(A-B)}\phi(\eta,\gamma,\delta,n) - [(1-\mu)+n\mu]z_0^{n-1}$$

or equivalently

$$|z|^{n-1} \leq \frac{(1-B)}{n^2(A-B)}\phi(\eta,\gamma,\delta,n),$$

which in turn implies the assertion of the theorem.

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