East Asian Math. J. 14 (1998), No. 1, pp. 99–106

A VERSION OF FERNIQUE LEMMA FOR GAUSSIAN PROCESSES

YONG-KAB CHOI* AND ZHENG-YAN LIN

ABSTRACT. We establish a version of Fernique lemma for Gaussian processes which plays an important role in studying their moduli of continuity properties and related limit theorems

1. Introduction and results

When one studies sample path continuity properties of Gaussian processes and sub-Gaussian processes, the Fernique lemma plays an important role (cf. e.g. Fernique ([4], [5]), Jain and Marcus [7]) Fernique type inequalities for not necessarily Gaussian processes have been also established by Csáki and Csörgő ([1], [2]) and Csáki et al. [3], etc.

For the sake of quoting one of them from Fernique ([4], [5]), let $X(t)$ be a separable centered Gaussian process on $[0,1]$ with $E\{X(t)\}^2 \leq \Gamma^2$, $\Gamma > 0$ and

$$E\{X(t) - X(s)\}^2 \leq \psi^2(|t-s|),$$

where ψ is assumed to be continuous and nondecreasing on $[0,1]$, and also such that $\int_1^\infty \psi(e^{-x^2})\,dx < \infty$. Then we have the following

PROPOSITION. (Fernique lemma) *Let $\{X(t),\ 0 \leq t \leq 1\}$ be as the above statements. Then for $x \geq 2$ we have*

$$P\Big\{\sup_{0\leq t\leq 1}|X(t)| > x\big(\Gamma + 4\int_1^\infty \psi(e^{-y^2})\,dy\big)\Big\} \leq C\int_x^\infty e^{-y^2/2}\,dy,$$

where C is a positive constant.

Our aim here is to obtain a version of Proposition A for a Gaussian process $\{X(t_1,\cdots,t_d) : a_i \leq t_i \leq b_i,\ i = 1,\cdots,d\}$ having several time parameters $t_1,\cdots,t_d$. Towards this end, let $\mathbb{D} = \{\mathbf{t} : \mathbf{t} =$

Received January 15, 1998

*Partially supported by BSRI Grant 97–1405, Ministry of Education, Korea

$(t_1,\cdots,t_d),\ a_i \le t_i \le b_i,\ i=1,\cdots,d\}$ be a real d-dimensional time parameter space with the usual Euclidean norm $\|\cdot\|$. Let $\{X(\mathbf{t}),\ \mathbf{t}\in\mathbb{D}\}$ be a real-valued separable Gaussian process with $EX(\mathbf{t})=0$. Suppose that

$$0<\Gamma^2 := \sup_{\mathbf{t}\in\mathbb{D}} E(X(\mathbf{t}))^2 < \infty, \quad \Gamma>0, \tag{1.1}$$

and

$$E\{X(\mathbf{t})-X(\mathbf{s})\}^2 \le \varphi^2(\|\mathbf{t}-\mathbf{s}\|), \tag{1.2}$$

where $\varphi(\cdot)$ is a nondecreasing continuous function such that

$$\int_0^\infty \varphi(e^{-y^2})\,dy<\infty.$$

Then we have the following theorem:

THEOREM 1. *Let $\{X(\mathbf{t}),\ \mathbf{t}\in\mathbb{D}\}$ be a centered Gaussian process satisfying conditions* (1.1) *and* (1.2). *Then, for $\lambda>0$, $x\ge 1$ and $\mathcal{A}>\sqrt{2d\log 2}$, we have*

$$P\Big\{\sup_{\mathbf{t}\in\mathbb{D}} X(\mathbf{t}) > x\{\Gamma + (2\sqrt{2}+2)\mathcal{A}\int_0^\infty \varphi(\sqrt{d}\lambda 2^{-y^2})\,dy\}\Big\}$$

$$\le (2^d+\mathcal{B})\Big(\prod_{i=1}^d\Big(\frac{b_i-a_i}{\lambda}\vee 1\Big)\Big)e^{-x^2/2},$$

where $a\vee b=\max\{a,b\}$ and

$$\mathcal{B}=\sum_{n=1}^\infty \exp\{-2^{n-1}(\mathcal{A}^2-2d\log 2)\}<\infty.$$

Theorem 1 is used as an essential lemma to obtain limit theorems on the increments of multi-parameter Gaussian processes. For instance, let us introduce moduli of continuity for the increments of a multi-parameter Gaussian process whose proof is based on Theorem 1. We define:

$$\mathbf{0}=(0,\cdots,0) \text{ and } \mathbf{1}=(1,\cdots,1) \text{ in } [0,1]^d;$$

$$\mathbf{t}\le\mathbf{s} \text{ iff } t_i\le s_i \quad \text{for all integers } 1\le i\le d;$$

$$\mathbf{h}=(h,\cdots,h)\in(0,1)^d.$$

THEOREM 2. [6] (moduli of continuity) *Let $\{X(\mathbf{t}),\ \mathbf{t} \in [0,1]^d\}$ be an almost surely continuous, centered d-parameter Gaussian process with $X(\mathbf{0}) = 0$ and stationary increments*

$$E\{X(\mathbf{t}) - X(\mathbf{s})\}^2 = \sigma^2(\|\mathbf{t} - \mathbf{s}\|), \quad \mathbf{t} \neq \mathbf{s} \in [0,1]^d,$$

where $\sigma(t)$ is a nondecreasing continuous, regularly varying function on $(0,1)$ with exponent γ for some $0 < \gamma < 1$ at zero. Assume also that there exist positive constants C_1 and C_2 such that, for $x > 0$,

$$\frac{d\sigma^2(x)}{dx} \leq C_1 \frac{\sigma^2(x)}{x} \qquad \text{and} \qquad \frac{d^2\sigma^2(x)}{dx^2} \leq C_2 \frac{\sigma^2(x)}{x^2}.$$

Then we have

$$\liminf_{h \to 0} \sup_{\mathbf{0} \leq \mathbf{t} \leq \mathbf{1} - \mathbf{h}} \frac{|X(\mathbf{t} + \mathbf{h}) - X(\mathbf{t})|}{\sigma(\sqrt{d}\, h)\sqrt{2d \log(1/h)}} = 1 \qquad \text{a.s.},$$

$$\lim_{h \to 0} \sup_{\mathbf{0} < \mathbf{s} \leq \mathbf{h}} \sup_{\mathbf{0} \leq \mathbf{t} \leq \mathbf{1} - \mathbf{s}} \frac{|X(\mathbf{t} + \mathbf{s}) - X(\mathbf{t})|}{\sigma(\sqrt{d}\, h)\sqrt{2d \log(1/h)}} = 1 \qquad \text{a.s.}$$

2. Proof of Theorem 1

For each $n = 0, 1, 2, \cdots$, set $\varepsilon_n = \lambda 2^{-2^n}$, $\lambda > 0$. For $\mathbf{k} = (k_1, \cdots, k_d)$, where $k_i = 0, 1, \cdots, k_{in} := [(b_i - a_i)/\varepsilon_n]$, $i = 1, \cdots, d$, define $\mathbf{t}_{\mathbf{k}}^{(n)} = (t_{1k_1}^{(n)}, \cdots, t_{dk_d}^{(n)})$ in $\mathbb{D}$, where

$$t_{ik_i}^{(n)} = a_i + k_i \varepsilon_n, \quad i = 1, \cdots, d.$$

Let

$$S_n = \{\mathbf{t}_{\mathbf{k}}^{(n)}, \mathbf{k} = \mathbf{0}, \cdots, \mathbf{k}_n := (k_{1n}, \cdots, k_{dn})\},$$

which contains $N_n := \prod_{i=1}^d k_{in}$ points, $N_n \leq 2^{2^n d} \prod_{i=1}^d (b_i - a_i)/\lambda$. Then the set $\cup_{n=0}^\infty S_n$ is dense in $\mathbb{D}$ and $S_n \subset S_{n+1}$. For $x \geq 1$ and $\mathcal{A} > \sqrt{2d \log 2}$, denote

$$x_m = x\, \mathcal{A}\, \varphi(\sqrt{d}\, \varepsilon_{m-1}) 2^{m/2}, \quad m \geq 1.$$

For $m \geq 1$, let $\delta_m = 2^{(m-1)/2}$. Then

$$2^{m/2} = 2(\sqrt{2}+1)(\delta_m - \delta_{m-1}).$$

Thus

$$\begin{aligned}
\sum_{m=1}^{\infty} x_m &= x\mathcal{A} \sum_{m=1}^{\infty} \varphi\big(\sqrt{d}\lambda 2^{-2^{m-1}}\big) 2^{m/2} \\
&= x\mathcal{A} \sum_{m=1}^{\infty} \varphi\big(\sqrt{d}\lambda 2^{-\delta_m{}^2}\big)(2\sqrt{2}+2)(\delta_m - \delta_{m-1}) \\
&\leq (2\sqrt{2}+2)x\mathcal{A} \sum_{m=1}^{\infty} \int_{\delta_{m-1}}^{\delta_m} \varphi\big(\sqrt{d}\lambda 2^{-y^2}\big)\, dy \\
&\leq (2\sqrt{2}+2)x\mathcal{A} \int_0^{\infty} \varphi\big(\sqrt{d}\lambda 2^{-y^2}\big)\, dy.
\end{aligned} \tag{2.1}$$

Therefore, by (2.1) we have

$$\begin{aligned}
&P\Big\{\sup_{\mathbf{t}\in\mathbb{D}} X(\mathbf{t}) > x\big(\Gamma + (2\sqrt{2}+2)\mathcal{A}\int_0^{\infty} \varphi\big(\sqrt{d}\lambda 2^{-y^2}\big)\, dy\big)\Big\} \\
&\leq P\Big\{\sup_{\mathbf{t}\in\mathbb{D}} X(\mathbf{t}) \geq x\Gamma + \sum_{m=1}^{\infty} x_m\Big\} \\
&= P\Big\{\max_{n\geq 0} \sup_{\mathbf{t}\in S_n} X(\mathbf{t}) \geq x\Gamma + \sum_{m=1}^{\infty} x_m\Big\} \\
&\leq \lim_{n\to\infty} P\Big\{\sup_{\mathbf{t}\in S_n} X(\mathbf{t}) \geq x\Gamma + \sum_{m=1}^{n} x_m\Big\}.
\end{aligned} \tag{2.2}$$

For $n \geq 0$, let

$$A_n = \Big\{\sup_{\mathbf{t}\in S_n} X(\mathbf{t}) \geq x\Gamma + \sum_{m=1}^{n} x_m\Big\}.$$

By induction we have

$$
\begin{aligned}
P(A_n) &= P(A_n \cap A_{n-1}) + P(A_n \cap A_{n-1}^c) \\
&\le P(A_{n-1}) + P(A_n \cap A_{n-1}^c) \\
&\le P(A_{n-2}) + P(A_{n-1} \cap A_{n-2}^c) + P(A_n \cap A_{n-1}^c) \\
&\le P(A_0) + \sum_{n=1}^{\infty} P(A_n \cap A_{n-1}^c) \\
&\le P(B_0) + \sum_{n=1}^{\infty} P(B_n \cap B_{n-1}^c),
\end{aligned}
\tag{2.3}
$$

where

$$
B_0 = \Big\{ \sup_{\mathbf{t} \in S_0} X(\mathbf{t}) \ge x\Gamma \Big\}, \quad B_n = \Big\{ \sup_{\mathbf{t} \in S_n} X(\mathbf{t}) \ge \sum_{m=1}^{n} x_m \Big\}, \quad n \ge 1.
$$

Now for $n \ge 1$, we have

$$
\begin{aligned}
& P(B_n \cap B_{n-1}^c) \\
= \;& P\Big\{ \big\{ \sup_{\mathbf{t} \in S_n} X(\mathbf{t}) \ge \sum_{m=1}^{n} x_m \big\} \cap \big\{ \sup_{\mathbf{s} \in S_{n-1}} X(\mathbf{s}) < \sum_{m=1}^{n-1} x_m \big\} \Big\} \\
\le \;& P\Big\{ \bigcup_{\mathbf{t} \in S_n} \big\{ X(\mathbf{t}) \ge \sum_{m=1}^{n} x_m \big\} \cap \bigcap_{\mathbf{s} \in S_{n-1}} \big\{ X(\mathbf{s}) < \sum_{m=1}^{n-1} x_m \big\} \Big\} \\
\le \;& P\Big\{ \bigcup_{\mathbf{t} \in S_n - S_{n-1}} \; \bigcup_{\substack{\mathbf{s} \in S_{n-1} \\ \|\mathbf{t}-\mathbf{s}\| \le \sqrt{d}\,\varepsilon_{n-1}}} \big\{ X(\mathbf{t}) - X(\mathbf{s}) \ge x_n \big\} \Big\} \\
\le \;& \sum_{\mathbf{t} \in S_n} \; \sum_{\substack{\mathbf{s} \in S_{n-1} \\ \|\mathbf{t}-\mathbf{s}\| \le \sqrt{d}\,\varepsilon_{n-1}}} P\big\{ X(\mathbf{t}) - X(\mathbf{s}) \ge x_n \big\}.
\end{aligned}
\tag{2.4}
$$

But by the assumption (1.2), we get

$$
E\{X(\mathbf{t}) - X(\mathbf{s})\}^2 \le \varphi^2(\|\mathbf{t} - \mathbf{s}\|) \le \varphi^2(\sqrt{d}\varepsilon_{n-1}), \quad n \ge 1. \tag{2.5}
$$

Hence, noting $\mathcal{A} > \sqrt{2d\log 2}$, $x \geq 1$ and the fact that there is only one point s in the set $\{\mathbf{s} \in S_{n-1} : \|\mathbf{t}-\mathbf{s}\| \leq \sqrt{d}\,\varepsilon_{n-1}\}$ for any $\mathbf{t} \in S_n - S_{n-1}$, it follows from (2.5) that (2.4) implies

$$\begin{aligned}
P\big(B_n \cap B_{n-1}^c\big) &\leq \sum_{\mathbf{t}\in S_n} \sum_{\substack{\mathbf{s}\in S_{n-1} \\ \|\mathbf{t}-\mathbf{s}\|\leq\sqrt{d}\,\varepsilon_{n-1}}} P\Big\{Z \geq \frac{x_n}{\varphi(\sqrt{d}\,\varepsilon_{n-1})}\Big\} \\
&\leq \Big(\prod_{i=1}^{d} \frac{b_i - a_i}{\lambda}\Big)(2^{2^n})^d P\Big\{Z \geq \frac{x_n}{\varphi(\sqrt{d}\,\varepsilon_{n-1})}\Big\} \\
&= 2^{2^n d}\Big(\prod_{i=1}^{d} \frac{b_i - a_i}{\lambda}\Big) P\Big\{Z \geq \mathcal{A}x2^{n/2}\Big\} \\
&\leq 2^{2^n d}\Big(\prod_{i=1}^{d} \frac{b_i - a_i}{\lambda}\Big)\frac{1}{2\sqrt{\pi}}e^{-\mathcal{A}^2x^22^{n-1}}
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2\sqrt{\pi}}\Big(\prod_{i=1}^{d} \frac{b_i - a_i}{\lambda}\Big)2^{2^n d}e^{-(\mathcal{A}^22^{n-1}-1/2)x^2}e^{-x^2/2} \\
&\leq \frac{1}{2\sqrt{\pi}}\Big(\prod_{i=1}^{d} \frac{b_i - a_i}{\lambda}\Big)e^{2^n d\log 2 - 2^{n-1}\mathcal{A}^2 + 1/2}e^{-x^2/2} \\
&= e^{-2^n((\mathcal{A}^2/2)-d\log 2)}\Big(\prod_{i=1}^{d}\Big(\frac{b_i - a_i}{\lambda} \vee 1\Big)\Big)e^{-x^2/2},
\end{aligned}$$

where Z denotes the standard normal random variable. In particular, if $\mathcal{A} > 0$ is such that

$$\frac{\mathcal{A}^2}{2} - d\log 2 > 0,$$

then

$$\sum_{n=1}^{\infty} P\big(B_n \cap B_{n-1}^c\big) \leq \mathcal{B}\Big(\prod_{i=1}^{d}\Big(\frac{b_i - a_i}{\lambda} \vee 1\Big)\Big)e^{-x^2/2}, \tag{2.6}$$

where

$$\mathcal{B} = \sum_{n=1}^{\infty} \exp\big\{-2^{n-1}(\mathcal{A}^2 - 2d\log 2)\big\} < \infty.$$

On the other hand,

$$
\begin{aligned}
P(B_0) &= P\Big\{\sup_{\mathbf{t}\in S_0} X(\mathbf{t}) \geq x\Gamma\Big\} \\
&\leq 2^d\Big(\prod_{i=1}^{d}\Big(\frac{b_i - a_i}{\lambda} \vee 1\Big)\Big)P\{Z \geq x\} \qquad (2.7)\\
&\leq 2^d\Big(\prod_{i=1}^{d}\Big(\frac{b_i - a_i}{\lambda} \vee 1\Big)\Big)e^{-x^2/2}.
\end{aligned}
$$

From (2.3), (2.6) and (2.7) we obtain, for any $n \geq 0$,

$$
P(A_n) \leq (2^d + \mathcal{B})\Big(\prod_{i=1}^{d}\Big(\frac{b_i - a_i}{\lambda} \vee 1\Big)\Big)e^{-x^2/2}. \qquad (2.8)
$$

Thus by (2.8) the inequality (2.2) gives to

$$
\begin{aligned}
&P\Big\{\sup_{\mathbf{t}\in\mathbb{D}} X(\mathbf{t}) > x\big(\Gamma + (2\sqrt{2}+2)\mathcal{A}\int_0^\infty \varphi\big(\sqrt{d}\lambda 2^{-y^2}\big)\,dy\big)\Big\} \\
&\leq (2^d + \mathcal{B})\Big(\prod_{i=1}^{d}\Big(\frac{b_i - a_i}{\lambda} \vee 1\Big)\Big)e^{-x^2/2}.
\end{aligned}
$$

This completes the proof of Theorem 1. □

References

[1] E. Csáki and M. Csörgő, *Fernique type inequalities for not necessarily Gaussian processes*, C. R. Math. Rep. Acad. Sci Canada **12** (1990), 149–154

[2] E. Csáki and M Csorgő, *Inequalities for increments of stochastic processes and moduli of continuity*, Ann. Probab. **20** (1992), 1031–1052.

[3] E. Csáki and M Csorgő and Q. M Shao, *Moduli of continuity for l^p-valued Gaussian processes*, In Tech. Rep. Ser. Lab. Res Stat. Probab. **160** (1991), Carleton University-University of Ottawa

[4] X. Fernique, *Continuite des processus and Gaussiens*, C R Acad Sci. Paris **258** (1964), 6058–6060

[5] X. Fernique, *Regularité des trajectoires des fonctions aléatoires Gaussiennes*, Lecture Notes in Mathematics **480** (1975), 1–96, Springer–Verlag, Berlin.

[6] K. S. Hwang, Y. K. Choi and J. S Kwon, *Moduli of continuity for the increments of a multi-parameter Gaussian process*, Submitted to J. Korean Math. Soc. 1998.

[7] N. C. Jain and M. B. Marcus, *Continuity of subgaussian processes*, In: Probability on Banach spaces (ed. J Kuelbs), Advances in Probability and Related Topics (series ed. P. Ney) 4 (1978), 81–196.

Department of Mathematics
College of Natural Science
Gyeongsang National University
Chinju 660–701, Korea

Department of Mathematics
Hangzhou University
Hangzhou 310028, People's Republic of China