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NONLINEAR SEMIGROUPS AND DIFFERENTIAL INCLUSIONS IN PROBABILISTIC NORMED SPACES

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ABSTRACT The purpose of this paper is to introduce and study the semigroups of nonlinear contractions in probabilistic normed spaces and to establish the Crandall-Liggett's exponential formula for some kind of accretive mappings in probabilistic normed spaces As applications, we utilize these results to study the Cauchy problem for a kind of differential inclusions with accertive mappings in probabilistic normed spaces.

1. Introduction

The concept of accretive mappings is of fundamental importance in the theory of set-valued nonlinear operators, differential equations and partial differential equations in Banach spaces, which was introduced independently by F. E. Browder ([3]) and T. Kato ([11]). On the other hand, many authors have done considerable works on semigroups of nonlinear contractions, differential equations and evolution equations in Banach spaces and Hilbert spaces ([1], [2], [4], [7], [8], [12], [13]).

Recently, the authors introduced the concept of accretive mappings ([5]) and some elementary properties of accretive mappings in probabilistic normed spaces have been deduced by K. S. Ha et al. ([9]).

The purpose of this paper is to introduce and study the semigroups of nonlinear contractions in probabilistic normed spaces and to prove

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that if A is an accretive mapping in probabilistic normed spaces satisfying the range condition, then A generates a semigroup of nonlinear contractions. As applications, we shall use these results to study the Cauchy problem of solutions for a kind of differential inclusions with accretive mappings in probabilistic normed spaces.

For the sake of convenience, we shall recall some definitions and notations ([5], [6], [16]).

Throughout this paper, we denote by $\mathcal{D}$ the set of distribution functions defined on $\mathbb{R}$, i.e., $f \in \mathcal{D}$ if f is nondecreasing left-continuous with $\sup_{t\in\mathbb{R}} f(t) = 1$ and $\inf_{t\in\mathbb{R}} f(t) = 0$.

DEFINITION 1.1. A *probabilistic normed space* (shortly, PN-space) is an ordered pair $(E, \mathcal{F})$, where E is a real linear space and $\mathcal{F}$ is a mapping from E into $\mathcal{D}$ (we denote $\mathcal{F}(x)$ by F_x) satisfying the following conditions: For all $x, y \in E$,

(PN-1) $F_x(t) = 1$ for all $t > 0$ if and only if $x = 0$;

(PN-2) $F_x(0) = 0$;

(PN-3) $F_{\alpha x}(t) = F_x(\frac{t}{|\alpha|})$ for any $\alpha \in \mathbb{R}$, $\alpha \neq 0$;

(PN-4) If $F_x(t_1) = 1$, $F_y(t_2) = 1$, then $F_{x+y}(t_1 + t_2) = 1$.

DEFINITION 1.2. A mapping $\Delta : [0,1] \times [0,1] \to [0,1]$ is called a *t-norm* if it satisfies the following conditions: For any $a, b, c, d \in [0,1]$,

(T-1) $\Delta(a, 1) = a$;

(T-2) $\Delta(a, b) = \Delta(b, a)$;

(T-3) $\Delta(c, d) \geq \Delta(a, b)$ for $c \geq a$ and $d \geq b$;

(T-4) $\Delta(\Delta(a, b), c) = \Delta(a, \Delta(b, c))$.

A *Menger PN-space* is a triple $(E, \mathcal{F}, \Delta)$, where $(E, \mathcal{F})$ is a PN-space and Δ is a t-norm satisfying

(PN-4′) $F_{x+y}(t_1+t_2) \geq \Delta(F_x(t_1), F_y(t_2))$ for all $x, y \in E$ and $t_1, t_2 \in \mathbb{R}^+ = [0, +\infty)$.

DEFINITION 1.3 ([5]). Let $(E, \mathcal{F}, \Delta)$ be a Menger PN-space.

(i) $A : \mathrm{D}(A) \subset E \to 2^E$ is called an *accretive mapping* if

$$F_{x-y}(t) \geq F_{x-y+\lambda(u-v)}(t)$$

for all $x, y \in \mathrm{D}(A)$, $u \in Ax$, $v \in Ay$ and $\lambda > 0$.

(ii) A is called a *maximal accretive mapping* if

$$F_{x-y_0}(t) \geq F_{x-y_0+\lambda(u-v_0)}(t)$$

for all $x \in \mathrm{D}(A)$, $u \in Ax$ and $\lambda > 0$, then $y_0 \in \mathrm{D}(A)$ and $v_0 \in Ay_0$.

(iii) A is called a *m-accretive mapping* if A is accretive and $I + A$ is surjective.

(iv) A is called a *strongly accretive mapping* if there exists a $k \in (0,1)$ such that

$$F_{(\lambda-k)(x-y)}(t) \geq F_{(\lambda-1)(x-y)+u-v}(t)$$

for all $x, y \in \mathrm{D}(A), u \in Ax, v \in Ay$ and $\lambda > k$.

(v) A is called a *dissipative mapping* (*maximal dissipative, m-dissipative, respectively*) if $-A$ is accretive (maximal accretive, m-accretive, respectively).

2. Semi-inner products in Menger PN-spaces

In this section, we always assume that $(E, \mathcal{F}, \Delta)$ is a Menger PN-space.

For any $\lambda \in (0,1]$, we define a real nonnegative function $P_\lambda : E \to \mathbb{R}^+$ as follows:

$$P_\lambda(x) = \inf\{t : F_x(t) > 1 - \lambda\} \text{ for all } x \in E.$$

From the definition of $P_\lambda(x)$, it is easy to prove the following:

PROPOSITION 2.1. *Let $(E, \mathcal{F}, \Delta)$ be a Menger PN-space with $\Delta(t,t) \geq t$ for all $t \in [0,1]$. Then for any $\lambda \in (0,1)$*

(i) *$P_\lambda(\alpha x) = |\alpha| P_\lambda(x)$ for all $\alpha \in \mathbb{R}$ and $x \in E$;*

(ii) *$P_\lambda(x+y) \leq P_\lambda(x) + P_\lambda(y)$ for all $x, y \in E$;*

(iii) *$(P_\lambda(x+ty) - P_\lambda(x))/t$ is nondecreasing in $t \in (0, +\infty)$ and $x, y \in E$;*

(iv) *$(P_\lambda(x) - P_\lambda(x-ty))/t$ is nonincreasing in $t \in (0, +\infty)$ and $x, y \in E$.*

It follows from Proposition 2.1 that the following limits exist:

$$\lim_{t\to 0^+} (P_\lambda(x+ty) - P_\lambda(x))/t \text{ and } \lim_{t\to 0^+} (P_\lambda(x) - P_\lambda(x-ty))/t.$$

In the sequel, we denote

$$[x,y]^+_\lambda = \lim_{t\to 0^+}(P_\lambda(x+ty)-P_\lambda(x))/t$$

and

$$[x,y]^-_\lambda = \lim_{t\to 0^+}(P_\lambda(x)-P_\lambda(x-ty))/t.$$

In what follows we give some basic properties of $[x,y]^\pm_\lambda$:

LEMMA 2.2. *Let $(E,\mathcal{F},\Delta)$ be a Menger PN-space with $\Delta(t,t)\ge t$ for all $t\in[0,1]$. Then we have the following :*

(i) $[x,y]^-_\lambda \le [x,y]^+_\lambda$;

(ii) $|[x,y]^\pm_\lambda| \le P_\lambda(y)$ *and* $[x,\alpha x]^\pm_\lambda = \alpha P_\lambda(x)$ *for all* $\alpha\in\mathbb{R}$;

(iii) $|[x,y]^\pm_\lambda - [x,z]^\pm_\lambda| \le P_\lambda(y-z)$;

(iv) $[x,y]^+_\lambda = -[x,-y]^-_\lambda = -[-x,y]^-_\lambda$;

(v) $[sx,ry]^\pm_\lambda = r[x,y]^\pm_\lambda$ *for all* $r,s\ge 0$;

(vi) $[x,y+z]^+_\lambda \le [x,y]^+_\lambda + [x,z]^+_\lambda$ *and* $[x,y+z]^-_\lambda \ge [x,y]^-_\lambda + [x,z]^-_\lambda$;

(vii) $[x,y+z]^+_\lambda \ge [x,y]^+_\lambda + [x,z]^-_\lambda$ *and* $[x,y+z]^-_\lambda \le [x,y]^-_\lambda + [x,z]^+_\lambda$;

(viii) $[x,y+\alpha x]^\pm_\lambda = [x,y]^\pm_\lambda + \alpha P_\lambda(x)$ *for all* $\alpha\in\mathbb{R}$;

(ix) *If* $x(t):[a,b]\to E$ *is differentiable in* $t\in(a,b)$ *and* $\varphi_\lambda(t) = P_\lambda(x(t))$, *then*

$$D^+\varphi_\lambda(t) = \lim_{h\to 0^+}(P_\lambda(x(t+h))-P_\lambda(x(t)))/h = [x(t),x'(t)]^+_\lambda;$$

$$D^-\varphi_\lambda(t) = \lim_{h\to 0^+}(P_\lambda(x(t))-P_\lambda(x(t-h)))/h = [x(t),x'(t)]^-_\lambda;$$

(x) $[x,y]^+_\lambda$ *is upper semi-continuous and* $[x,y]^-_\lambda$ *is lower semi-continuous.*

Proof. Properties (i)-(v) follow easily and so the details are omitted here.

(vi) Since

$$\begin{aligned}&(P_\lambda(x+t(y+z))-P_\lambda(x))/t\\ &\le \frac{1}{2t}\{[P_\lambda(x+2ty)-P_\lambda(x)]+[P_\lambda(x+2tz)-P_\lambda(x)]\},\end{aligned}$$

we have

$$[x,y+z]^+_\lambda \le [x,y]^+_\lambda + [x,z]^+_\lambda.$$

Similarly, we can prove that $[x, y+z]_\lambda^- \geq [x,y]_\lambda^- + [x,z]_\lambda^-$.
(vii) Since

$$[x,y]_\lambda^+ = [x, y+z-z]_\lambda^+ \leq [x,y+z]_\lambda^+ + [x,-z]_\lambda^+,$$

from (iv), it follows that $[x,-z]_\lambda^+ = -[x,z]_\lambda^-$ and so we have

$$[x,y+z]_\lambda^+ \geq [x,y]_\lambda^+ + [x,z]_\lambda^-.$$

(viii) By (vi) and (vii), we have

$$[x, y+\alpha x]_\lambda^+ \leq [x,y]_\lambda^+ + [x,\alpha x]_\lambda^+ = [x,y]_\lambda^+ + \alpha P_\lambda(x)$$

and

$$[x, y+\alpha x]_\lambda^+ \geq [x,y]_\lambda^+ + [x,\alpha x]_\lambda^- = [x,y]_\lambda^+ + \alpha P_\lambda(x),$$

respectively. Therefore, we have

$$[x,y+\alpha x]_\lambda^+ = [x,y]_\lambda^+ + \alpha P_\lambda(x).$$

Similarly, we can prove that $[x, y+\alpha x]_\lambda^- = [x,y]_\lambda^- + \alpha P_\lambda(x)$.
(ix) Since

$$\begin{aligned}
&|D^+\varphi_\lambda(t) - [x(t), x'(t)]_\lambda^+| \\
&= |\lim_{h\to 0^+}(P_\lambda(x(t+h)) - P_\lambda(x(t)))/h \\
&\quad - \lim_{h\to 0^+}(P_\lambda(x(t)+hx'(t)) - P_\lambda(x(t)))/h| \\
&= |\lim_{h\to 0^+}\frac{1}{h}(P_\lambda(x(t+h)) - P_\lambda(x(t)+hx'(t)))| \\
&\leq \lim_{h\to 0^+}|\frac{1}{h}(P_\lambda(x(t+h)-x(t)-hx'(t)))| \\
&= \lim_{h\to 0^+}|P_\lambda(\frac{x(t+h)-x(t)-hx'(t)}{h})| = 0,
\end{aligned}$$

(ix) is true.
(x) Letting $x_n \to x$ and $y_n \to y$, since

$$[x_n, y_n]_\lambda^+ \leq \frac{1}{t}(P_\lambda(x_n+ty_n) - P_\lambda(x_n)) \text{ for all } t>0,$$

we have

$$\overline{\lim_{n\to\infty}}[x_n, y_n]^+_\lambda \leq \frac{1}{t}(P_\lambda(x+ty) - P_\lambda(x)).$$

Letting $t \to 0^+$, it follows that $\overline{\lim}_{n\to\infty}[x_n, y_n]^+_\lambda \leq [x,y]^+_\lambda$, which means that $[x,y]^+_\lambda$ is upper semi-continuous.

Similarly, we can prove that $[x,y]^-_\lambda$ is lower semi-continuous. This completes the proof.

Next, we define a mapping $j_\lambda : E \to 2^{E^*}$ (E^* is the dual space of E) by

$$j_\lambda(x) = \{f_\lambda \in E^* : f_\lambda(x) = P_\lambda(x),\ [x,y]^-_\lambda \leq f_\lambda(y) \leq [x,y]^+_\lambda,\ y \in E\}.$$

Now we claim that for any $x \in E$, $j_\lambda(x) \neq \emptyset$. In fact, for any $y_0 \in E$, we define $f_\lambda(\alpha y_0) = \alpha[x, y_0]^+_\lambda$ for all $\alpha \in \mathbb{R}$.

(a) If $\alpha \geq 0$, then $f_\lambda(\alpha y_0) = [x, \alpha y_0]^+_\lambda$;

(b) If $\alpha < 0$, then

$$\begin{aligned}\alpha[x,y_0]^+_\lambda &= -|\alpha|[x,y_0]^+_\lambda = -[x,|\alpha|y_0]^+_\lambda \\ &= [x,-|\alpha|y_0]^-_\lambda = [x,\alpha y_0]^-_\lambda \\ &\leq [x,\alpha y_0]^+_\lambda.\end{aligned}$$

Therefore, we have $f_\lambda(\alpha y_0) \leq [x, \alpha y_0]^+_\lambda$ for all $\alpha \in \mathbb{R}$. By (v) and (vi) of Lemma 2.2, $[x,y]^+_\lambda$ is subadditive in $y \in E$. By using the Hahn-Banach Theorem ([15]), there exists a linear functional $\widetilde{f_\lambda} : E \to \mathbb{R}$ such that $\widetilde{f_\lambda}(\alpha y_0) = f_\lambda(\alpha y_0)$ and $-[x,-y]^+_\lambda \leq \widetilde{f_\lambda}(y) \leq [x,y]^+_\lambda$ for all $y \in E$, i.e.,

$$[x,y]^-_\lambda \leq \widetilde{f_\lambda}(y) \leq [x,y]^+_\lambda.$$

Especially, we have $\widetilde{f_\lambda}(x) = [x,x]^+_\lambda = P_\lambda(x)$.

The continuity of $\widetilde{f_\lambda}$ follows from $|\widetilde{f_\lambda}(x)| \leq |[x,y]^+_\lambda| \leq P_\lambda(y)$ immediately. Therefore, we know $\widetilde{f_\lambda} \in j_\lambda(x)$. This completes the proof.

Moreover, we can also prove that $j_\lambda(x)$ is convex. Hence, by the Banach-Alaoglu Theorem, we have the following :

PROPOSITION 2.3. *For each $x \in E$ and $\lambda \in (0,1]$, $j_\lambda(x)$ is a nonempty convex weak* compact subset of E^*.*

In view of the above argument and Proposition 2.3, we have the following :

PROPOSITION 2.4. $[x,y]_\lambda^+ = \max\{f_\lambda(y) : f_\lambda \in j_\lambda(x)\}$ *and*

$$[x,y]_\lambda^- = \min\{f_\lambda(y) : f_\lambda \in j_\lambda(x)\}.$$

DEFINITION 2.1. (i) $(x,y)_\lambda^+ = P_\lambda(x) \cdot [x,y]_\lambda^+$ is called the *upper semi-inner product* with respect to $\lambda \in (0,1]$,

(ii) $(x,y)_\lambda^- = P_\lambda(x) \cdot [x,y]_\lambda^-$ is called the *lower semi-inner product* with respect to $\lambda \in (0,1]$.

For some properties of the semi-inner products, refer to [14].

DEFINITION 2.2. The mapping $\Im_\lambda : E \to 2^{E^*}$ defined by

$$\Im_\lambda(x) = \{P_\lambda(x) \cdot f_\lambda : f_\lambda \in j_\lambda(x)\} \text{ for all } x \in E$$

is called the *duality mapping* with respect to $\lambda \in (0,1]$.

It follows from Lemma 2.2 that the following corollary holds:

COROLLARY 2.5. (i) $(x,y)_\lambda^- \le (x,y)_\lambda^+$;
(ii) $|(x,y)_\lambda^\pm| \le P_\lambda(x) \cdot P_\lambda(y)$ *and* $(x,\alpha x)_\lambda^\pm \le \alpha P_\lambda^2(x)$ *for all* $\alpha \in \mathbb{R}$;
(iii) $|(x,y)_\lambda^\pm - (x,z)_\lambda^\pm| \le P_\lambda(x) \cdot P_\lambda(y-z)$;
(iv) $(x,y)_\lambda^+ = (-x,-y)_\lambda^- = -(-x,y)_\lambda^-$;
(v) $(sx,ry)_\lambda^\pm = s \cdot r \cdot (x,y)_\lambda^\pm$ *for all* $r,s \ge 0$;
(vi) $(x,y+z)_\lambda^+ \le (x,y)_\lambda^+ + (x,z)_\lambda^+$ and $(x,y+z)_\lambda^- \ge (x,y)_\lambda^- + (x,z)_\lambda^-$;
(vii) $(x,y+z)_\lambda^+ \ge (x,y)_\lambda^+ + (x,z)_\lambda^-$ and $(x,y+z)_\lambda^- \le (x,y)_\lambda^- + (x,z)_\lambda^+$;
(viii) $(x,y+\alpha x)_\lambda^\pm = (x,y)_\lambda^\pm + \alpha P_\lambda^2(x)$ *for all* $\alpha \in \mathbb{R}$;
(ix) *If* $x(t) : [a,b] \to E$ *is differentiable in* $t \in (a,b)$ *and* $\varphi_\lambda(t) = P_\lambda^2(x(t))$, *then*

$$D^+\varphi_\lambda(t) = 2(x(t),x'(t))_\lambda^+ \text{ and } D^-\varphi_\lambda(t) = 2(x(t),x'(t))_\lambda^-;$$

(x) $(x,y)_\lambda^+$ *is upper semi-continuous and* $(x,y)_\lambda^-$ *is lower semi-continuous.*

3. Accretive mappings and nonlinear semigroups in PN-spaces

In this section, we always assume that $(E, \mathcal{F}, \Delta)$ is a complete Menger PN-space with $\Delta(t,t) \geq t$ for all $t \in [0,1]$.

LEMMA 3.1. *Let $A : \mathrm{D}(A) \subset E \to 2^E$ be a mapping. Then the following conclusions are equivalent:*

(i) *A is accretive;*

(ii) *$P_\lambda(x-y) \leq P_\lambda(x-y+\epsilon(u-v))$ for all $x, y \in \mathrm{D}(A)$, $u \in Ax$, $v \in Ay$ and for all $\epsilon > 0$, $\lambda \in (0,1]$;*

(iii) *$[x-y, u-v]^+_\lambda \geq 0$ for all $x, y \in \mathrm{D}(A)$, $u \in Ax$, $v \in Ay$ and $\lambda \in (0,1]$.*

Proof. (i) $\Longleftrightarrow$ (ii). If A is accretive, then

$$F_{x-y}(t) \geq F_{x-y+\epsilon(u-v)}(t)$$

for all $x, y \in \mathrm{D}(A)$, $u \in Ax$, $v \in Ay$ and $\epsilon > 0$. Besides, for given $x, y \in \mathrm{D}(A)$, $u \in Ax$, $v \in Ay$ and $\epsilon > 0$, letting

$$\begin{aligned} P_\lambda(x-y+\epsilon(u-v)) &= \inf\{t : F_{x-y+\epsilon(u-v)}(t) > 1-\lambda\} \\ &= \lim_{n\to\infty}\{t_n : F_{x-y+\epsilon(u-v)}(t_n) > 1-\lambda\}, \end{aligned}$$

then we have $F_{x-y}(t_n) > 1-\lambda$ for all $n \geq 1$ and so

$$P_\lambda(x-y) = \inf\{t : F_{x-y}(t) > 1-\lambda\} \leq \lim_{n\to\infty} t_n,$$

which implies that the conclusion (ii) is true.

Conversely, suppose that (ii) is true, but the conclusion (i) is not true. Then there exist $x_0, y_0 \in \mathrm{D}(A)$, $\epsilon_0 > 0$, $u_0 \in Ax_0$, $v_0 \in Ay_0$ and $t_0 > 0$ such taht

$$F_{x_0-y_0}(t_0) < F_{x_0-y_0+\epsilon_0(u_0-v_0)}(t_0).$$

Therefore, there exists $\lambda_0 \in (0,1]$ such that $F_{x_0-y_0}(t_0) = 1-\lambda_0$. This implies that

$$P_{\lambda_0}(x_0-y_0) = \inf\{t : F_{x_0-y_0}(t) > 1-\lambda_0\} \geq t_0.$$

Since $F_{x_0-y_0+\epsilon_0(u_0-v_0)}(t_0) > 1-\lambda_0$ and $F_{x_0-y_0+\epsilon_0(u_0-v_0)}(t_0)$ is left continuous, there exists $\delta_0 > 0$ such that

$$F_{x_0-y_0+\epsilon_0(u_0-v_0)}(t_0-\delta_0) > 1-\lambda_0.$$

Hence we have

$$P_{\lambda_0}(x_0-y_0+\epsilon_0(u_0-v_0)) \le t_0-\delta_0 < t_0 \le P_{\lambda_0}(x_0-y_0),$$

which is a contradiction

(ii) $\Longleftrightarrow$ (iii) By Proposition 2.1 (iii) and the definition of $[\cdot,\cdot]^+_\lambda$, it is obvious that the conclusions are true. This completes the proof.

LEMMA 3.2. *Let $A : \mathrm{D}(A) \subset E \to 2^E$ be an accretive mapping and $J_\epsilon = (I+\epsilon A)^{-1}$ for all $\epsilon > 0$, then*

(i) *$P_\lambda(J_\epsilon x - J_\epsilon y) \le P_\lambda(x-y)$ and $F_{J_\epsilon x - J_\epsilon y}(t) \ge F_{x-y}(t)$ for all $t > 0$, $\lambda \in (0,1]$, and $x, y \in \mathrm{R}(I+\epsilon A)$, the range of $I+\epsilon A$;*

(ii) *$P_\lambda(J^n_\epsilon x - x) \le n \cdot P_\lambda(J_\epsilon x - x)$ for all $\lambda \in (0,1]$, an integer $n > 0$ and $x \in \mathrm{R}((I+\epsilon A)^n)$, and*

$$F_{J^n_\epsilon x - x}(t) \ge F_{J_\epsilon x - x}(\frac{t}{n}) \text{ for all } t > 0 \text{ and } x \in \mathrm{R}((I+\epsilon A)^n);$$

(iii) *If $x_j \in \mathrm{R}(I+\epsilon A)$ and $x_j \to x_0 \in \mathrm{D}(A) \cap \mathrm{R}(I+\epsilon A)$, then*

$$\overline{\lim_{j\to\infty}} P_\lambda(J_\epsilon x_j - x_j) \le \epsilon \cdot \inf_{u \in Ax_0} P_\lambda(u) \text{ for all } \quad \lambda \in (0,1]$$

and

$$\underline{\lim_{j\to\infty}} F_{J_\epsilon x_j - x_j}(t) \ge \sup_{u \in Ax_0} F_u(\frac{t}{\epsilon}) \text{ for all } t > 0.$$

Proof. (i) is an immediate consequence of Lemma 3.1 and the accretivity of A.

(ii) can be obtained from (i) immediately

Next, we prove (iii). For any given $u \in Ax_0$, letting $w = x_0 + \epsilon u$, then we have

$$x_0 = (I+\epsilon A)^{-1} w = J_\epsilon w$$

and

$$P_\lambda(J_\epsilon x_j - x_j) \leq P_\lambda(J_\epsilon x_j - J_\epsilon w) + P_\lambda(J_\epsilon w - x_j).$$

Hence it follows that

$$\begin{aligned}\overline{\lim_{j\to\infty}} P_\lambda(J_\epsilon x_j - x_j) &\leq \overline{\lim_{j\to\infty}} (P_\lambda(x_j - w) + P_\lambda(x_0 - x_j)) \\ &\leq \overline{\lim_{j\to\infty}} P_\lambda(x_j - w) \\ &\leq \overline{\lim_{j\to\infty}} (P_\lambda(x_j - x_0) + P_\lambda(x_0 - w)) \\ &\leq P_\lambda(-\epsilon u) = \epsilon P_\lambda(u).\end{aligned}$$

Therefore, by the arbitrariness of $u \in Ax_0$, we have

$$\overline{\lim_{j\to\infty}} P_\lambda(J_\epsilon x_j - x_j) \leq \epsilon \cdot \inf_{u\in Ax_0} P_\lambda(u).$$

On the other hand, since

$$\begin{aligned}F_{J_\epsilon x_j - x_j}(t) &\geq \Delta(F_{J_\epsilon x_j - J_\epsilon w}(t - \frac{\eta}{2}), F_{J_\epsilon w - x_j}(\frac{\eta}{2})) \\ &\geq \Delta(F_{x_j - w}(t - \frac{\eta}{2}), F_{x_0 - x_j}(\frac{\eta}{2}))\end{aligned}$$

and

$$F_{x_j - w}(t - \frac{\eta}{2}) \geq \Delta(F_{x_j - x_0}(\frac{\eta}{2}), F_{\epsilon u}(t - \eta))$$

for all $\eta < t$, we have

$$F_{J_\epsilon x_j - x_j}(t) \geq \Delta(F_{\epsilon u}(t - \eta), F_{x_0 - x_j}(\frac{\eta}{2}))$$

and so

$$\underline{\lim_{j\to\infty}} F_{J_\epsilon x_j - x_j}(t) \geq F_u(\frac{t-\eta}{\epsilon}).$$

Since $F_u(t)$ is left-continuous, letting $\eta \to 0^+$, we have

$$\underline{\lim_{j\to\infty}} F_{J_\epsilon x_j - x_j}(t) \geq F_u(\frac{t}{\epsilon}),$$

which implies that

$$\varliminf_{j\to\infty} F_{J_\epsilon x_j - x_j}(t) \geq \sup_{u\in Ax_0} F_u(\frac{t}{\epsilon}).$$

This completes the proof.

We are now in a position to consider the Cauchy problem of the following differential inclusion with an accretive mapping A:

$$\text{(E3.1)} \qquad \begin{cases} u'(t) \in -Au(t), \ t > 0, \\ u(0) = u_0 \in \mathrm{D}(A). \end{cases}$$

DEFINITION 3.1. A function $u(\cdot) \in C(\mathbb{R}^+, E)$ is called a *strong solution* of (E3.1) if it satisfies the following conditions :

(i) $u(0) = u_0$;

(ii) There exists $y \in E$ such that

$$F_{u(t)-u(s)}(k) \geq F_{(t-s)y}(k) \text{ for all } k > 0 \text{ and } t, s \in \mathbb{R}^+$$

(In this case, we also say $u(\cdot)$ to be *Lipschitz continuous*);

(iii) The derivative $u'(t)$ of $u(\cdot)$ exists and satisfies

$$u'(t) \in -Au(t) \text{ for almost all } t \in (0, +\infty).$$

Thus, we have the following:

THEOREM 3.3. *Let $(E, \mathcal{F}, \Delta)$ be a complete Menger PN-space with $\Delta(t,t) \geq t$ for all $t \in [0,1]$ and $A : \mathrm{D}(A) \subset E \to 2^E$ be an accretive mapping. Then (E3.1) has at most one strong solution.*

Proof. Let $u(\cdot)$ and $v(\cdot)$ be two strong solutions of (E3.1) and denote $\varphi_\lambda(t) = P_\lambda(u(t) - v(t))$ for all $\lambda \in (0,1]$. Then, by Lemma 2.2 (ix), we have

$$D^-\varphi_\lambda(t) = [u(t) - v(t), u'(t) - v'(t)]_\lambda^-.$$

Therefore, there exist $w(t) \in Au(t)$ and $z(t) \in Av(t)$ such that

$$u'(t) = -w(t), \quad v'(t) = -z(t) \text{ for almost all } t \in (0, +\infty)$$

and so we have

$$\begin{aligned} D^- \varphi_\lambda(t) &= [u(t) - v(t), (w(t) - z(t))]_\lambda^- \\ &= -[u(t) - v(t), w(t) - z(t)]_\lambda^+ \\ &\le 0. \end{aligned}$$

Therefore, we have

$$P_\lambda(u(t) - v(t)) \le P_\lambda(u(0) - v(0)) = 0 \text{ for all } \lambda \in (0, 1].$$

If $u(t_0) - v(t_0) \ne 0$ for some $t_0 \in \mathbb{R}^+$, then there exists $k_0 > 0$ such that

$$F_{u(t_0)-v(t_0)}(k_0) < 1.$$

Letting $F_{u(t_0)-v(t_0)}(k_0) = 1 - \lambda_0$, then $\lambda_0 \in (0, 1]$ and so

$$P_{\lambda_0}(u(t_0) - v(t_0)) = \inf\{k : F_{u(t_0)-v(t_0)}(k) > 1 - \lambda_0\} \ge k_0 > 0,$$

which contradicts $P_{\lambda_0}(u(t_0) - v(t_0)) = 0$. This implies that $u(t) = v(t)$ for all $t \in \mathbb{R}^+$. This completes the proof.

DEFINITION 3.2. Let $(E, \mathcal{F}, \Delta)$ be a complete Menger PN-space and C be a closed subset of E. A family of operators, $\{T(t) : C \to E : t \ge 0\}$, is called a *semigroup of nonlinear contractions* if it satisfies the following conditions :

(i) $T(0)x = x$ for all $x \in C$;
(ii) $T(t)T(s) = T(t + s)$ for all $t, s \ge 0$;
(iii) The mapping $t \mapsto T(t)x$ is continuous for any $x \in C$;
(iv) $F_{T(t)x-T(t)y}(k) \ge F_{x-y}(k)$ for all $x, y \in C$, $t \ge 0$ and $k > 0$.

THEOREM 3.4. *Let $A : \mathrm{D}(A) \subset E \to 2^E$ be an accretive mapping satisfying the following conditions:*

$$(I + \epsilon A)(\mathrm{D}(A)) \supset \overline{\mathrm{D}(A)}, \text{ the closure of } D(A), \text{ for all } \epsilon > 0.$$

Then for any $x \in \overline{\mathrm{D}(A)}$, the following limit exists

$$T(t)x = \lim_{\epsilon \to 0^+} (I + \epsilon A)^{-[\frac{t}{\epsilon}]} x \text{ for all } t \ge 0,$$

where $[\frac{t}{\epsilon}]$ is the largest integer which does not exceed $\frac{t}{\epsilon}$. Moreover, $\{T(t) : t \ge 0\}$ is a semigroup of nonlinear contractions.

In order to prove Theorem 3.4, we need the following:

LEMMA 3.5. *Let $A : \mathrm{D}(A) \subset E \to 2^E$ be an accretive mapping and $\overline{\mathrm{D}(A)} \subset (I + \epsilon A)(\mathrm{D}(A))$ for all $\epsilon > 0$. Then*

$$F_{J_\epsilon^m x - J_\mu^n x}(t) \geq \sup_{u \in Ax} F_u(t \cdot ((m\epsilon - n\mu)^2 + m\epsilon^2 + n\mu^2)^{-\frac{1}{2}})$$

for all $x \in \mathrm{D}(A)$, ϵ, $\mu > 0$ and m, n are nonnegative integers.

Proof. We first prove that for any $x \in \mathrm{D}(A)$, ϵ, $\mu > 0$ and $\lambda \in (0, 1]$,

$$(3.1) \quad P_\lambda(J_\epsilon^m x - J_\mu^n x) \leq \{(m\epsilon - n\mu)^2 + m\epsilon^2 + n\mu^2\}^{\frac{1}{2}} \cdot \inf_{u \in Ax} P_\lambda(u),$$

where m, n are nonnegative integers.

For each $x \in \mathrm{D}(A)$, ϵ, $\mu > 0$ and $\lambda \in (0, 1]$, let

$$P_{m,n} = P_\lambda(J_\epsilon^m x - J_\mu^n x), \quad m, n = 0, 1, 2, \cdots.$$

By (ii) and (iii) of Lemma 3.2, we have

$$P_{m,0} \leq m\epsilon \cdot \inf_{u \in Ax} P_\lambda(u), \quad m = 0, 1, 2, \cdots,$$
$$P_{0,n} \leq n\mu \cdot \inf_{u \in Ax} P_\lambda(u), \quad n = 0, 1, 2, \cdots.$$

These mean that (3.1) holds for $n = 0$ or $m = 0$.

Now we suppose that (3.1) holds for a couple of integers $(m - 1, n)$, $(m, n - 1)$. For $x \in \mathrm{D}(J_\epsilon)$ and $y \in \mathrm{D}(J_\mu)$, setting $\delta = \frac{\epsilon\mu}{\epsilon + \mu}$, we can easily check

$$J_\delta(\frac{\mu}{\epsilon + \mu} x + \frac{\epsilon}{\epsilon + \mu} J_\epsilon x) = J_\epsilon x,$$
$$J_\delta(\frac{\epsilon}{\epsilon + \mu} y + \frac{\mu}{\epsilon + \mu} J_\mu y) = J_\mu y.$$

Therefore, we have

$$\begin{aligned}
&P_{m,n} \\
&= P_\lambda(J_\epsilon \cdot J_\epsilon^{m-1} x - J_\mu \cdot J_\mu^{n-1} x) \\
&= P_\lambda(J_{\frac{\epsilon\mu}{\epsilon+\mu}}(\frac{\mu}{\epsilon + \mu} J_\epsilon^{m-1} x + \frac{\epsilon}{\epsilon + \mu} J_\epsilon^m x) \\
&\quad - J_{\frac{\epsilon\mu}{\epsilon+\mu}}(\frac{\epsilon}{\epsilon + \mu} J_\mu^{n-1} x + \frac{\mu}{\epsilon + \mu} J_\mu^n x)) \\
&\leq P_\lambda(\frac{\mu}{\epsilon + \mu} J_\epsilon^{m-1} x + \frac{\epsilon}{\epsilon + \mu} J_\epsilon^m x - \frac{\epsilon}{\epsilon + \mu} J_\mu^{n-1} x - \frac{\mu}{\epsilon + \mu} J_\mu^n x) \\
&\leq \frac{\epsilon}{\epsilon + \mu} P_\lambda(J_\epsilon^m x - J_\mu^{n-1} x) + \frac{\mu}{\epsilon + \mu} P_\lambda(J_\epsilon^{m-1} x - J_\mu^n x),
\end{aligned}$$

i.e.,

$$P_{m,n} \le \frac{\epsilon}{\epsilon+\mu} P_{m,n-1} + \frac{\mu}{\epsilon+\mu} P_{m-1,n}$$

and thus we have

$$\begin{aligned}
&P_{m,n} \\
&\le \frac{\epsilon}{\epsilon+\mu}\{(m\epsilon - n\mu)^2 + 2\mu(m\epsilon - n\mu) + m\epsilon^2 + n\mu^2\}^{\frac{1}{2}} \cdot \inf_{u\in Ax} P_\lambda(u) \\
&\quad + \frac{\mu}{\epsilon+\mu}\{(m\epsilon - n\mu)^2 - 2\epsilon(m\epsilon - n\mu) + m\epsilon^2 + n\mu^2\}^{\frac{1}{2}} \cdot \inf_{u\in Ax} P_\lambda(u) \\
&\le \{\frac{\epsilon}{\epsilon+\mu}[(m\epsilon - n\mu)^2 + 2\mu(m\epsilon - n\mu) + m\epsilon^2 + n\mu^2] \\
&\quad + \frac{\mu}{\epsilon+\mu}[(m\epsilon - n\mu)^2 - 2\epsilon(m\epsilon - n\mu) + m\epsilon^2 + n\mu^2]\}^{\frac{1}{2}} \cdot \inf_{u\in Ax} P_\lambda(u) \\
&= \{(m\epsilon - n\mu)^2 + m\epsilon^2 + n\mu^2\}^{\frac{1}{2}} \cdot \inf_{u\in Ax} P_\lambda(u).
\end{aligned}$$

Therefore, the conclusion of (3.1) is proved.

Now, suppose that the conclusion of Lemma 3.5 is not true. There exist x_0, m_0, n_0, ϵ_0, μ_0 and $t_0 > 0$ such that

$$F_{J_{\epsilon_0}^{m_0}x_0 - J_{\mu_0}^{n_0}x_0}(t_0) < \sup_{u\in Ax_0} F_u(t_0 \cdot \{(m_0\epsilon_0 - n_0\mu_0)^2 + m_0\epsilon_0^2 + n_0\mu_0^2\}^{-\frac{1}{2}}).$$

Therefore, there exists $u_0 \in Ax_0$ such that

$$F_{J_{\epsilon_0}^{m_0}x_0 - J_{u_0}^{n_0}x_0}(t_0) < F_{u_0}(t_0 \cdot \{(m_0\epsilon_0 - n_0\mu_0)^2 + m_0\epsilon_0^2 + n_0\mu_0^2\}^{-\frac{1}{2}}).$$

Letting $F_{J_{\epsilon_0}^{m_0}x_0 - J_{\mu_0}^{n_0}x_0}(t_0) = 1 - \lambda_0$, then $\lambda_0 \in (0, 1]$. It is obvious that

$$P_{\lambda_0}(J_{\epsilon_0}^{m_0}x_0 - J_{\mu_0}^{n_0}x_0) = \inf\{t : F_{J_{\epsilon_0}^{m_0}x_0 - J_{\mu_0}^{n_0}x_0}(t) > 1 - \lambda_0\} \ge t_0$$

and

$$\begin{aligned}
P_{\lambda_0}(u_0) &= \inf\{t : F_{u_0}(t) > 1 - \lambda_0\} \\
&< t_0 \cdot \{(m_0\epsilon_0 - n_0\mu_0)^2 + m_0\epsilon_0^2 + n_0\mu_0^2\}^{-\frac{1}{2}}.
\end{aligned}$$

Hence we have

$$P_{\lambda_0}(J_{\epsilon_0}^{m_0}x_0 - J_{\mu_0}^{n_0}x_0) > \{(m_0\epsilon_0 - n_0\mu_0)^2 + m_0\epsilon_0^2 + n_0\mu_0^2\}^{\frac{1}{2}} \cdot \inf_{u\in Ax} P_{\lambda_0}(u),$$

which contradicts (3.1). This completes the proof.

Proof of Theorem 3.4. For each $x \in \mathrm{D}(A)$, by Lemma 3.5, we have

$$F_{J_\epsilon^{[\frac{t}{\epsilon}]}x - J_\mu^{[\frac{t}{\mu}]}x}(k) \geq \sup_{u \in Ax} F_u(k \cdot \{([\frac{t}{\epsilon}] \cdot \epsilon - [\frac{t}{\mu}] \cdot \mu)^2 + [\frac{t}{\epsilon}] \cdot \epsilon^2 + [\frac{t}{\mu}] \cdot \mu^2\}^{-\frac{1}{2}}.$$

Since

$$\{([\frac{t}{\epsilon}] \cdot \epsilon - [\frac{t}{\mu}] \cdot \mu)^2 + [\frac{t}{\epsilon}] \cdot \epsilon^2 + [\frac{t}{\mu}] \cdot \mu^2\}^{\frac{1}{2}} \leq \{(\epsilon + \mu)^2 + (\epsilon + \mu)t\}^{\frac{1}{2}},$$

it follows that

$$F_{J_\epsilon^{[\frac{t}{\epsilon}]}x - J_\mu^{[\frac{t}{\mu}]}x}(k) \geq \sup_{u \in Ax} F_u(k \cdot \{(\epsilon + \mu)^2 + (\epsilon + \mu)t\}^{-\frac{1}{2}}).$$

Letting ϵ, $\mu \to 0^+$, we have

$$\lim_{\epsilon, \mu \to 0^+} F_{J_\epsilon^{[\frac{t}{\epsilon}]} - J_\mu^{[\frac{t}{\mu}]}x}(k) = 1 \text{ for all } k > 0.$$

This implies that $\{J_\epsilon^{[\frac{t}{\epsilon}]}x\}$ is a Cauchy sequence in E. Hence the limit

$$T(t)x = \lim_{\epsilon \to 0^+} J_\epsilon^{[\frac{t}{\epsilon}]}x \tag{3.2}$$

exists. Since $J_\epsilon^{[\frac{t}{\epsilon}]}$ is contractive, for each $x \in \overline{\mathrm{D}(A)}$ the limit in (3.2) still exists and $T(t)$ is contractive on $\overline{\mathrm{D}(A)}$ for all $t \geq 0$.

Next, let $t, s \geq 0$ and $x \in \mathrm{D}(A)$. Then, by Lemma 3.5, we have

$$F_{J_\epsilon^{[\frac{t}{\epsilon}]}x - J_\epsilon^{[\frac{s}{\epsilon}]}x}(k) \geq \sup_{u \in Ax} F_u(k \cdot \{([\frac{t}{\epsilon}] \cdot \epsilon - [\frac{s}{\epsilon}] \cdot \epsilon)^2 + [\frac{t}{\epsilon}] \cdot \epsilon^2 + [\frac{s}{\epsilon}] \cdot \epsilon^2\}^{-\frac{1}{2}}).$$

Since

$$([\frac{t}{\epsilon}] \cdot \epsilon - [\frac{s}{\epsilon}] \cdot \epsilon)^2 + [\frac{t}{\epsilon}] \cdot \epsilon^2 + [\frac{s}{\epsilon}] \cdot \epsilon^2 \leq (|t - s| + \epsilon)^2 + (t + s) \cdot \epsilon,$$

for any $u \in Ax$ and $k > 0$ we have

$$\begin{aligned} F_{J_\epsilon^{[\frac{t}{\epsilon}]}x - J_\epsilon^{[\frac{s}{\epsilon}]}x}(k) &\geq \sup_{u \in Ax} F_u(k \cdot \{(|t - s| + \epsilon)^2 + (t + s)\epsilon\}^{-\frac{1}{2}}) \\ &\geq F_u(k \cdot \{(|t - s| + \epsilon)^2 + (t + s) \cdot \epsilon\}^{-\frac{1}{2}}) \end{aligned} \tag{3.3}$$

and

$$\begin{aligned}
&F_{T(t)x-T(s)x}(k) \\
&\geq \Delta(F_{T(t)x-J_\epsilon^{[\frac{t}{\epsilon}]}x}(\frac{\eta}{3}), F_{J_\epsilon^{[\frac{t}{\epsilon}]}x-T(s)x}(k-\frac{\eta}{3})) \\
&\geq \Delta(F_{T(t)x-J_\epsilon^{[\frac{t}{\epsilon}]}x}(\frac{\eta}{3}), \Delta(F_{J_\epsilon^{[\frac{t}{\epsilon}]}x-J_\epsilon^{[\frac{s}{\epsilon}]}x}(k-\frac{2\eta}{3}), F_{J_\epsilon^{[\frac{s}{\epsilon}]}x-T(s)x}(\frac{\eta}{3}))),
\end{aligned}$$

where $0<\eta<k$. Since

$$\lim_{\epsilon\to 0^+} F_{T(t)x-J_\epsilon^{[\frac{t}{\epsilon}]}x}(\frac{\eta}{3}) = 1 \text{ and } \quad \lim_{\epsilon\to 0^+} F_{J_\epsilon^{[\frac{s}{\epsilon}]}x-T(s)x}(\frac{\eta}{3}) = 1,$$

letting $\epsilon \to 0^+$, we have

$$F_{T(t)x-T(s)x}(k) \geq \varliminf_{\epsilon\to 0^+} F_{J_\epsilon^{[\frac{t}{\epsilon}]}x-J_\epsilon^{[\frac{s}{\epsilon}]}x}(k-\frac{2\eta}{3}) \tag{3.4}$$

for all $0<\eta<k$ and $k>0$. By (3.3) and the left-continuity of $F_u(\cdot)$, we have

$$\varliminf_{\epsilon\to 0^+} F_{J_\epsilon^{[\frac{t}{\epsilon}]}x-J_\epsilon^{[\frac{s}{\epsilon}]}x}(k-\frac{2\eta}{3}) \geq F_u((k-\frac{2\eta}{3})\cdot|t-s|^{-1}) \tag{3.5}$$

for all $\eta\in(0,k)$ and $u\in Ax$. By (3.4) and (3.5), we have

$$F_{T(t)x-T(s)x}(k) \geq F_u((k-\frac{2\eta}{3})\cdot|t-s|^{-1})$$

for all $\eta\in(0,k)$ and $u\in Ax$. Letting $\eta\to 0^+$, by the left-continuity of $F_u(\cdot)$, we have

$$F_{T(t)x-T(s)x}(k) \geq F_u(\frac{k}{|t-s|}) \text{ for all } u\in Ax.$$

This shows that $T(t)x$ is a Lipschitz continuous function in t for any $x\in \mathrm{D}(A)$. Since $T(t)$ is contractive, $T(t)x$ is a continuous function in t for any $x\in\overline{\mathrm{D}(A)}$.

Finally, letting $x \in \mathrm{D}(A)$ and $t, s \geq 0$, then

$$F_{J_\epsilon^{[\frac{t+s}{\epsilon}]}x - J_\epsilon^{[\frac{t}{\epsilon}]}J_\epsilon^{[\frac{s}{\epsilon}]}x}(k) \geq \sup_{u \in Ax} F_u(k \cdot \{(([\frac{t+s}{\epsilon}] \cdot \epsilon - ([\frac{t}{\epsilon}] + [\frac{s}{\epsilon}]) \cdot \epsilon)^2 + [\frac{t+s}{\epsilon}] \cdot \epsilon^2 + ([\frac{t}{\epsilon}] + [\frac{s}{\epsilon}])\epsilon^2\}^{-\frac{1}{2}})$$
$$\geq \sup_{u \in Ax} F_u(k \cdot \{(3\epsilon)^2 + 2(t+s)\epsilon\}^{-\frac{1}{2}})$$

for all $k > 0$. Letting $\epsilon \to 0^+$, we have

$$\lim_{\epsilon \to 0^+} F_{J_\epsilon^{[\frac{t+s}{\epsilon}]}x - J_\epsilon^{[\frac{t}{\epsilon}]} \cdot J_\epsilon^{[\frac{s}{\epsilon}]}x}(k) = 1 \text{ for all } k > 0,$$

which implies that $T(t+s)x = T(t)T(s)x$ for all $t, s \geq 0$ and $x \in \mathrm{D}(A)$. Therefore, since $T(t)$ is a contraction, it follows that

$$T(t+s)x = T(t) \cdot T(s)x \text{ for all } x \in \overline{\mathrm{D}(A)} \text{ and } t, s \geq 0.$$

This completes the proof.

REMARK. Theorem 3.4 is a generalization of the Crandall-Liggett's exponential formula for some kind of accretive mappings in Banach spaces to probabilistic normed spaces.

THEOREM 3.5. *Let $A \cdot E \to 2^E$ be an accretive mapping satisfying the following conditions :*

(i) $\overline{\mathrm{D}(A)} \subset \mathrm{R}(I + \epsilon A)$ *for all* $\epsilon > 0$;

(ii) *If* $x_n \in \mathrm{D}(A)$, $y_n \in Ax_n$, $x_n \to x$ *and* $y_n \to y$ *as* $n \to \infty$, *then* $x \in \mathrm{D}(A)$ *and* $y \in Ax$.

Let $\{T(t) : t \geq 0\}$ be the semigroups generated by A as given in Theorem 3.4. If $x \in \mathrm{D}(A)$ and $u(t) = T(t)x$ is strongly differentiable for almost all $t > 0$, then $u(t)$ is the unique strong solution of the Cauchy problem (E3.1):

To prove Theorem 3.5, we need the following:

LEMMA 3.6. *Let $A : D(A) \subset E \to 2^E$ be an accretive mapping satisfying $\mathrm{D}(A) \subset \mathrm{R}(I + \epsilon A)$ for all $\epsilon > 0$ and $\{T(t) : t \geq 0\}$ be the semigroup given in Theorem 3.4. If $x \in \mathrm{D}(A)$, then for any $x_0 \in \mathrm{D}(A)$, $y_0 \in Ax_0$, $t \geq 0$ and $\lambda \in (0, 1]$,*

$$P_\lambda(T(t)x - x_0) \leq P_\lambda(x - x_0) + \int_0^t [T(s)x - x_0, -y_0]_\lambda^+ ds.$$

Proof. Let $x \in D(A)$, $x_0 \in D(A)$ and $y_0 \in Ax_0$. For any $\epsilon > 0$ and positive integer N, we have

$$\epsilon^{-1}(J_\epsilon^N x - J_\epsilon^{N-1} x) \in -AJ_\epsilon^N x.$$

Since A is accretive, by Lemma 3.1, we have

$$\begin{aligned}[J_\epsilon^N x - x_0, \frac{1}{\epsilon}(J_\epsilon^N x - J_\epsilon^{N-1} x) + y_0]_\lambda^- \\ = -[J_\epsilon^N x - x_0, \frac{1}{\epsilon}(J_\epsilon^{N-1} x - J_\epsilon^N x) - y_0]_\lambda^+ \le 0.\end{aligned} \tag{3.6}$$

By Lemma 2.2 (vi), we have

$$\begin{aligned}&J_\epsilon^N x - x_0, \frac{1}{\epsilon}(J_\epsilon^N x - J_\epsilon^{N-1} x) + y_0]_\epsilon^- \\ &\ge [J_\epsilon^N x - x_0, \frac{1}{\epsilon}(J_\epsilon^N x - J_\epsilon^{N-1} x)]_\lambda^- + [J_\epsilon^N x - x_0, y_0]_\lambda^-.\end{aligned}$$

In view of Proposition 2.1 (iv), we have

$$\begin{aligned}&[J_\epsilon^N x - x_0, \frac{1}{\epsilon}(J_\epsilon^N x - J_\epsilon^{N-1} x) + y_0]_\lambda^- \\ &\ge \frac{1}{\epsilon}(P_\lambda(J_\epsilon^N x - x_0) - P_\lambda(J_\epsilon^N x - x_0 - (J_\epsilon^N x - J_\epsilon^{N-1} x))) \\ &\quad + [J_\epsilon^N x - x_0, y_0]_\lambda^-.\end{aligned} \tag{3.7}$$

By (3.6) and (3.7), we have

$$P_\lambda(J_\epsilon^N x - x_0) \le P_\lambda(J_\epsilon^{N-1} x - x_0) + \epsilon[J_\epsilon^N x - x_0, -y_0]_\lambda^+. \tag{3.8}$$

Adding up the inequalities in (3.8) from $N = 1$ to $N = n$, we have

$$P_\lambda(J_\epsilon^n x - x_0) \le P_\lambda(x - x_0) + \sum_{N=1}^{n} \epsilon[J_\epsilon^N x - x_0, -y_0]_\lambda^+. \tag{3.9}$$

Letting $t \ge 0$ and $n = [\frac{t}{\epsilon}]$, then (3.9) can be written as follows:

$$P_\lambda(J_\epsilon^{[\frac{t}{\epsilon}]} x - x_0) \le P_\lambda(x - x_0) + \int_\epsilon^{([\frac{t}{\epsilon}]+1)\epsilon} [J_\epsilon^{[\frac{s}{\epsilon}]} x - x_0, -y_0]_\lambda^+ ds.$$

Since $|[J_\epsilon^{[\frac{s}{\epsilon}]}x - x_0, -y_0]_\lambda^+| \le P_\lambda(y_0)$, letting $\epsilon \to 0^+$, by the Lebesgue's convergence theorem, it follows from the upper semi-continuity of $[\cdot,\cdot]_\lambda^+$ that

$$P_\lambda(T(t)x - x_0) \le P_\lambda(x - x_0) + \int_0^t \overline{\lim_{\epsilon\to 0}}[J_\epsilon^{[\frac{s}{\epsilon}]}x - x_0, -y_0]_\lambda^+ ds$$
$$\le P_\lambda(x - x_0) + \int_0^t [T(s)x - x_0, -y_0]_\lambda^+ ds.$$

This completes the proof.

Proof of Theorem 3.5. For $x \in D(A)$, if $T(t)x$ has a derivative $\frac{d}{dt}T(t)x|_{t=t_0} = y$ at $t = t_0 > 0$, then, by Lemma 3.6, we have

$$P_\lambda(T(t_0 + h)x - x_0) \le P_\lambda(T(t_0)x - x_0) + \int_0^h [T(t_0 + s)x - x_0, -y_0]_\lambda^+ ds$$

for all $h > 0$. Dividing by $h > 0$ on both sides and letting $h \to 0^+$, from Lemma 2.2 (ix), we have

$$[T(t_0)x - x_0, y]_\lambda^+ \le [T(t_0)x - x_0, -y_0]_\lambda^+.$$

It follows from Lemma 2.2 (vii) that

$$\begin{aligned} &[T(t_0)x - x_0, y + y_0]_\lambda^- \\ &\le [T(t_0)x - x_0, y]_\lambda^+ + [T(t_0)x - x_0, y_0]_\lambda^- \\ &= [T(t_0)x - x_0, y]_\lambda^+ - [T(t_0)x - x_0, -y_0]_\lambda^+ \\ &\le 0. \end{aligned} \tag{3.10}$$

By the condition (i), for any $\epsilon \in (0, t_0)$, there exist $x_\epsilon \in D(A)$ and $y_\epsilon \in Ax_\epsilon$ such that

$$x_\epsilon + \epsilon y_\epsilon = T(t_0 - \epsilon)x.$$

Taking $x_0 = x_\epsilon$, $y_0 = y_\epsilon = \epsilon^{-1}(T(t_0 - \epsilon)x - x_\epsilon)$ in (3.10), we have

$$\begin{aligned} 0 &\ge [T(t_0)x - x_\epsilon, y + \epsilon^{-1}(T(t_0 - \epsilon)x - x_\epsilon)]_\lambda^- \\ &= [T(t_0)x - x_\epsilon, y + \epsilon^{-1}(T(t_0 - \epsilon)x - T(t_0)x) + \epsilon^{-1}(T(t_0)x - x_\epsilon)]_\lambda^- \\ &= \epsilon^{-1}P_\lambda(T(t_0)x - x_\epsilon) \\ &\quad + [T(t_0)x - x_\epsilon, y + \epsilon^{-1}(T(t_0 - \epsilon)x - T(t_0)x)]_\lambda^- \\ &\ge \epsilon^{-1}P_\lambda(T(t_0)x - x_\epsilon) - P_\lambda(y + \epsilon^{-1}(T(t_0 - \epsilon)x - T(t_0)x)), \end{aligned}$$

i.e.,

$$P_\lambda(T(t_0)x - x_\epsilon) \leq P_\lambda(\epsilon y + (T(t_0-\epsilon)x - T(t_0)x)) \text{ for all } \lambda \in (0,1].$$

Therefore, we must have

$$F_{T(t_0)x - x_\epsilon}(k) \geq F_{\epsilon y + T(t_0-\epsilon)x - T(t_0)x}(k) \text{ for all } k \geq 0 \tag{3.11}$$

and so $x_\epsilon \to T(t_0)x$ as $\epsilon \to 0^+$. Since

$$\begin{aligned} &F_{y+y_\epsilon}(k) \\ &= F_{y-\epsilon^{-1}(T(t_0)x - T(t_0-\epsilon)x) + \epsilon^{-1}(T(t_0)x - x_\epsilon)}(k) \\ &\geq \Delta(F_{y-\epsilon^{-1}(T(t_0)x - T(t_0-\epsilon)x)}(\frac{k}{2}), F_{\epsilon^{-1}(T(t_0)x - x_\epsilon)}(\frac{k}{2})), \end{aligned} \tag{3.12}$$

from (3.11), (3.12) and $\lim_{\epsilon\to 0^+} \epsilon^{-1}(T(t_0)x - T(t_0-\epsilon)x) = y$, it follows that

$$F_{y+y_\epsilon}(k) \geq F_{y-\epsilon^{-1}(T(t_0)x - T(t_0-\epsilon)x)}(\frac{k}{2}) \to 1 \text{ as } \epsilon \to 0^+$$

and so $y_\epsilon \to -y$ as $\epsilon \to 0^+$. By the condition (ii), we have $T(t_0)x \in$ D(A) and $y \in -AT(t_0)x$. This completes the proof.

4. An open question

In the end of this paper, we suggest the following open question :

Let $(E, \mathcal{F}, \Delta)$ be a complete Menger PN-space and $A : E \to 2^E$ be a continuous accretive mapping. Then is A a m-accretive mapping ?

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