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THE HP–VERSION OF THE FINITE ELEMENT METHOD UNDER NUMERICAL QUADRATURE RULES

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ABSTRACT. we consider the hp–version to solve non-constant coefficients elliptic equations $-\mathrm{div}(a\nabla u) = f$ with Dirichlet boundary conditions on a bounded polygonal domain Ω in R^2 In [6], M Suri obtained an optimal error-estimate for the hp–version. $\|u - u_p^h\|_{1,\Omega} \leq Cp^{-(\sigma-1)}h^{min(p,\sigma-1)}\|u\|_{\sigma,\Omega}$ This optimal result follows under the assumption that all integrations are performed exactly In practice, the integrals are seldom computed exactly. The numerical quadrature rule scheme is needed to compute the integrals in the variational formulation of the discrete problem In this paper we consider a family $G_p = \{I_m\}$ of numerical quadrature rules satisfying certain properties, which can be used for calculating the integrals Under the numerical quadrature rules we will give the variational form of our non-constant coefficients elliptic problem and derive an error estimate of $\|u - \tilde{u}_p^h\|_{1,\Omega}$.

1. Introduction

The finite element method is a particular kind of Ritz-Galerkin procedure in which the approximating finite-dimensional subspaces are composed of piecewise polynomials defined on a partition of the given domain. The convergence is obtained by increasing the dimension of these subspaces in some manner. There are three versions of the finite element method. The h-version is the traditional approach obtained by fixing the degree p of the piecewise polynomials at some value (usually $p = 1, 2, 3$) and refining the mesh in order to achieve convergence. The p-version, in contrast, fixes the mesh and achieves

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the accuracy by increasing the degree p uniformly or selectively. The $hp-$version is the combination of both.

In this paper, to solve non-constant coefficients elliptic equations with Dirichlet boundary conditions on a bounded polygonal domain Ω in R^2 we consider the $hp-$version with a quasi-uniform mesh and uniform p. In [6], I. Babuška and M. Suri already obtained the following optimal estimate for the $hp-$version:

$$(1.1) \qquad \begin{aligned} &\|u - u_p^h\|_{1,\Omega} \\ &\leq C\, p^{-(\sigma-1)} h^{min(p,\sigma-1)} \|u\|_{k,\Omega} \quad \text{for all } u \in H_0^\sigma(\Omega),\ \sigma \geq 1, \end{aligned}$$

where C is independent of u, h, and p [but depends on Ω and σ].

The above optimal result follows under the assumption that all integrations are performed exactly. In practice, the integrals are seldom computed exactly. The numerical quadrature rule scheme is needed to compute the integrals in the variational formulation of the discrete problem. Thus we first consider a family $G_p = \{I_m\}$ of numerical quadrature rules satisfying certain properties, which can be used for calculating the integrals in the stiffness matrix of (2.17). Then, under the numerical quadrature rules we will give the variational form of our non-constant coefficients elliptic problem and derive an error estimate of $\|u - \tilde{u}_p^h\|_{1,\Omega}$ where $\tilde{u}_p^h$ is an approximation satisfying (3.6). We also analyze the cases in which the overintegration may improve the accuracy of the approximation to allow for optimal results.

2. Preliminaries

Let Ω be a closed and bounded polygonal domain in R^2 with the boundary Γ. Let $\mathcal{M} = \{\mathcal{J}^h\}, h \geq 0$ be a quasi-uniform, regular family of meshes $\mathcal{J}^h = \{\Omega_k^h\}$ defined on Ω, where Ω_k^h is a closed quadrilateral, and

$$(2.1) \qquad \max_{\Omega^h \in \mathcal{J}^h} \operatorname{diam}(\Omega^h) = h \quad \text{for all } \Omega^h, \mathcal{J}^h \in \mathcal{M}.$$

Further we assume that for each $\Omega_k^h \in \mathcal{J}^h$ there exists an invertible mapping $T_k^h : \widehat{\Omega} \to \Omega_k^h$ with the following correspondence:

$$\widehat{x} \in \widehat{\Omega} \longleftrightarrow x = T_k^h(\widehat{x}) \in \Omega_k^h, \tag{2.2}$$

$$\widehat{t} \in U_p(\widehat{\Omega}) \longleftrightarrow t = \widehat{t} \circ (T_k^h)^{-1} \in U_p(\Omega_k^h), \tag{2.3}$$

where $\widehat{\Omega}$ denotes the reference elements $\widehat{I}^2 = [-1,1]^2$ in R^2 ,

$$\begin{aligned} &U_p(\widehat{\Omega}) \\ &= \{\, \widehat{t} : \widehat{t} \text{ is a polynomial of degree} \le p \text{ in each variable on } \widehat{\Omega} \,\} \end{aligned} \tag{2.4}$$

and

$$U_p(\Omega_k^h) = \{\, t : \widehat{t} = t \circ T_k^h \in U_p(\widehat{\Omega}) \,\} \tag{2.5}$$

We now consider the following model problem of elliptic equations :

Find $u \in H_0^1(\Omega)$, such that

$$-\mathrm{div}(a \nabla u) = f \quad \text{in} \quad \Omega \subset R^2, \tag{2.6}$$

where two functions a and f satisfy a compatibility condition to ensure a solution exists, and

$$H_0^1(\Omega) = \{u \in H^1(\Omega) \,:\, u \text{ vanishes on } \Gamma\}. \tag{2.7}$$

For the sake of simplicity, we assume that

$$0 < A_1 \le a(x) \le A_2 \quad \text{for all} \quad x \in \Omega, \text{and} \tag{2.8}$$

$$f \in L_2(\Omega). \tag{2.9}$$

In addition, we also assume that there exists a constant $M \ge 1$ such that

$$\|T_k^h\|_{m,\infty,\widehat{\Omega}}\,, \quad \|(T_k^h)^{-1}\|_{m,\infty,\Omega_k^h} \le A \quad \text{for} \quad 0 \le m \le M, \tag{2.10}$$

$$\|\widehat{J}_k^h\|_{m,\infty,\widehat{\Omega}}\,, \quad \|(\widehat{J}_k^h)^{-1}\|_{m,\infty,\Omega_k^h} \le A \quad \text{for} \quad 0 \le m \le M-1, \tag{2.11}$$

where $\widehat{J}_k^h$ and $(\widehat{J}_k^h)^{-1}$ denote the Jacobians of T_k^h and $(T_k^h)^{-1}$ respectively.

Then, as seen in [8,theorem 3.1.2], we obtain the following correspondence: For any $\alpha \in [1,\infty]$, $0 \le m \le M$,

$$\widehat{t} \in W^{m,\alpha}(\widehat{\Omega}) \longleftrightarrow t = \widehat{t} \circ (T_k^h)^{-1} \in W^{m,\alpha}(\Omega_k^h) \tag{2.12}$$

with norm equivalence

$$C_1 h^{(m-\frac{2}{\alpha})} \|t\|_{m,\alpha,\Omega_k^h} \le \|\widehat{t}\|_{m,\alpha,\widehat{\Omega}} \le C_2 h^{(m-\frac{2}{\alpha})} \|t\|_{m,\alpha,\Omega_k^h} \tag{2.13}$$

with the subscript α omitted when $\alpha = 2$. Namely, we have

$$C_1 h^{(m-1)} \|t\|_{m,\Omega_k^h} \leq \|\widehat{t}\|_{m,\widehat{\Omega}} \leq C_2 h^{(m-1)} \|t\|_{m,\Omega_k^h}. \tag{2.14}$$

Let us define

(2.15) $S_p^h(\Omega) = \{u \in H^1(\Omega) \,:\, u_{\Omega_k^h} \circ (T_k^h) \in U_p(\widehat{\Omega}) \text{ for all } \Omega_k^h \in \mathcal{J}^h\}$,

where $u_{\Omega_k^h}$ denotes the restriction of $u \in H^1(\Omega)$ to $\Omega_k^h \in \mathcal{J}^h$, and

$$S_{p,0}^h(\Omega) = S_p^h(\Omega) \cap H_0^1(\Omega). \tag{2.16}$$

Then, using the hp–version of the finite element method with the mesh $\mathcal{J}^h = \{\Omega_k^h\}$ we obtain the following discrete variational form of (2.6):

Find $\quad u_p^h \in S_{p,0}^h(\Omega) \quad$ satisfying

$$B(u_p^h, v_p^h) = (f, v_p^h)_\Omega \quad \text{for all} \quad v_p^h \in S_{p,0}^h(\Omega), \tag{2.17}$$

where

$$B(u,v) = \int_\Omega a \nabla u \cdot \nabla v \, dx, \tag{2.18}$$

the usual inner product

$$(f,v)_\Omega = \int_\Omega f v \, dx. \tag{2.19}$$

Let us now give some approximation results which will be used later.

LEMMA 2.1. *For each integer $l \geq 0$, there exists a sequence of projections*

$$\Pi_p^l \,:\, H^l(\widehat{\Omega}) \to U_p(\widehat{\Omega}), \quad p = 1,2,3,\cdots \quad \textit{such that}$$

(2.20) $\quad \Pi_p^l \widehat{v}_p = \widehat{v}_p \quad$ *for all* $\quad \widehat{v}_p \in U_p(\widehat{\Omega})$,

(2.21) $\quad \|\widehat{u} - \Pi_p^l \widehat{u}\|_{s,\widehat{\Omega}} \leq C\, p^{-(r-s)} \|\widehat{u}\|_{r,\widehat{\Omega}} \quad$ *for all* $\quad \widehat{u} \in H^r(\widehat{\Omega})$ *with* $\quad 0 \leq s \leq l \leq r$.

Proof. See [9, *Lemma* 3.1].

LEMMA 2.2. *Suppose that $T_k^h : \widehat{\Omega} \longrightarrow \Omega_k^h$ is an invertible affine mapping. Then for any $u \in H^\sigma(\Omega), \sigma \geq 0$ we have*

$$(2.22) \qquad \inf_{\widehat{v} \in U_p(\widehat{\Omega})} \|\widehat{u}_{\Omega_k^h} - \widehat{v}\|_{\sigma,\widehat{\Omega}} \leq Ch^\mu \|u_{\Omega_k^h}\|_{\sigma,\Omega_k^h},$$

where $\mu = \min(p, \sigma - 1)$ and C is independent of h, p and u.

Proof. The proof is given in [6].

LEMMA 2.3. *For each $u \in H^\sigma(\Omega)$ and $\Omega_k^h \in \mathcal{J}^h$ there exists a sequence $z_p^h \in U_p(\Omega_k^h), p = 1, 2, \cdots$ such that for any $0 \leq r \leq \sigma$*

$$(2.23) \qquad \|u_{\Omega_k^h} - z_p^h\|_{r,\Omega_k^h} \leq Ch^{(\mu - r + 1)} p^{-(\sigma - r)} \|u_{\Omega_k^h}\|_{\sigma,\Omega_k^h} \text{ for all } \Omega_k^h \in \mathcal{J}^h,$$

where $\mu = \min(p, \sigma - 1)$ and C is independent of h, p and u.

Proof. See [6, Lemma 4.5].

Let $u \in H^\sigma(\Omega), \sigma > 1$ be the solution of (2.6) and u_p^h the hp-version finite element solution of (2.17). Then, as seen in [6] we have an estimate

$$(2.24) \qquad \|u - u_p^h\|_{1,\Omega} \leq Ch^{\min(p,\sigma-1)} p^{-(\sigma-1)} \|u\|_{\sigma,\Omega},$$

where C is independent of u, h and p.

3. The hp-version under numerical quadrature rules

We consider numerical quadrature rules I_m defined on the reference element $\widehat{\Omega}$ by

$$(3.1) \qquad I_m(\widehat{f}) = \sum_{i=1}^{n(m)} \widehat{w}_i^m \widehat{f}(\widehat{x}_i^m) \sim \int_{\widehat{\Omega}} \widehat{f}(\widehat{x})\, d\widehat{x},$$

where m is a positive integer. Let $G_p = \{I_m\}$ be a family of quadrature rules I_m with respect to $U_p(\widehat{\Omega})$, $p = 1, 2, 3, \cdots$, satisfying the following properties: For each $I_m \in G_p$,

(K1) $\widehat{w}_i^m > 0$ and $\widehat{x}_i^m \in \widehat{\Omega}$ for $i = 1, \cdots, n(m)$.

(K2) $I_m(\widehat{f}^2) \le C_1 \|\widehat{f}\|_{0,\widehat{\Omega}}^2$ for all $\widehat{f} \in U_p(\widehat{\Omega})$.

(K3) $C_2 \|\widetilde{f}\|_{0,\widehat{\Omega}}^2 \le I_m(\widetilde{f}^2)$ for all $\widetilde{f} \in \widetilde{U}_p(\widehat{\Omega})$,

where $\widetilde{U}_p(\widehat{\Omega}) = \{ \dfrac{\partial \widehat{f}}{\partial \widehat{x}_i} : \widehat{f} \in U_p(\widehat{\Omega}) \} \subset U_p(\widehat{\Omega})$.

(K4) $I_m(\widehat{f}) = \displaystyle\int_{\widehat{\Omega}} \widehat{f}(\widehat{x})\, d\widehat{x}$ for all $\widehat{f} \in U_{d(m)}(\widehat{\Omega})$,

where $d(m) \ge \widetilde{d}(p) > 0$.

We also get a family $G_{p,\Omega} = \{I_{m,\Omega}\}$ of numerical quadrature rules with respect to $S_p^h(\Omega)$, defined by

$$
(3.2)\quad I_{m,\Omega_k^h}(f_{\Omega_k^h}) = \sum_{j=1}^{n(m)} w_j^k f_{\Omega_k^h}(x_j^m) = \sum_{j=1}^{n(m)} \widehat{w}_j^m \widehat{J}_k^h(\widehat{x}_j^m)(f_{\Omega_k^h} \circ T_k^h)(\widehat{x}_j^m)
$$
$$
= I_m(\widehat{J}_k^h \widehat{f}_{\Omega_k^h})
$$

and

$$
(3.3)\qquad I_{m,\Omega}(f) = \sum_{\Omega_k^h \in \mathcal{J}^h} I_{m,\Omega_k^h}(f_{\Omega_k^h}).
$$

In particular, one may be interested in Gauss-Legendre(G-L) quadrature rules. Let L_q denote the cross-products of $q-$point G-L rules along the $\widehat{x}_1$ and $\widehat{x}_2$ axes on $\widehat{\Omega} = \widehat{I} \times \widehat{I}$, given by

$$
L_q(\widehat{f}) = \sum_{i=1}^{q} \sum_{j=1}^{q} \widehat{w}_i^q \widehat{w}_j^q \widehat{f}(\widehat{x}_{ij}^q) \quad \text{for all} \quad \widehat{f} \in L_2(\widehat{\Omega}),
$$

where $\widehat{x}_{ij}^q = (\widehat{x}_i^q, \widehat{x}_j^q) \in \widehat{\Omega} = \widehat{I} \times \widehat{I}$ with the weights $\widehat{w}_i^q$ and $\widehat{w}_j^q$.

We consider a family $\{L_q\}_{q \ge l(p)}$ of G-L quadrature rules with respect to $U_p(\widehat{\Omega})$ such that $l(p) = p+1$. Then, $\{L_q\}_{q \ge l(p)}$ satisfy the properties $(K1)-(K4)$. In fact, when $q \ge p+1$ $L_q(\widehat{f})$ is exact for all $\widehat{f} \in U_{d(q)}(\widehat{\Omega})$ with $d(q) \ge 2p+1 > 0$, so that (K2) and (K3) hold with $C_1 = C_2 = 1$.

Now, we denote by DF the 2×2 Jacobian matrix of $F : R^2 \to R^2$, and define two discrete inner products

$$(3.4) \qquad (u,v)_{m,\Omega_k^h} = I_{m,\Omega_k^h}((uv)_{\Omega_k^h}) = I_m(\widehat{J}_k^h \widehat{(uv)}_{\Omega_k^h}) \text{ on } \Omega_k^h \in \mathcal{J}^h,$$

$$(3.5) \qquad (u,v)_{m,\Omega} = \sum_{\Omega_k^h \in \mathcal{J}^h} (u,v)_{m,\Omega_k^h} \text{ on } \Omega.$$

Then, under numerical quadrature rules I_m in G_p we obtain the following actual problem of (2.17): Find $\widetilde{u}_p^h \in S_{p,0}^h(\Omega)$, such that

$$(3.6) \qquad B_{m,\Omega}(\widetilde{u}_p^h, v_p^h) = (f, v_p^h)_\Omega \text{ for all } v_p^h \in S_{p,0}^h(\Omega),$$

where

$$(3.7) \quad B_{m,\Omega}(\widetilde{u}_p^h, v_p^h) = \sum_{\Omega_k^h \in \mathcal{J}^h} I_{m,\Omega_k^h}(a\nabla\widetilde{u}_p^h \cdot \nabla v_p^h)$$

$$= \sum_{\Omega_k^h \in \mathcal{J}^h} I_m(\widehat{J}_k^h \widehat{a}(\nabla\widehat{\widetilde{u}}_p^h \widehat{D(T_k^h)^{-1}})(\nabla\widehat{v}_p^h \widehat{D(T_k^h)^{-1}})^t)$$

$$= \sum_{\Omega_k^h \in \mathcal{J}^h} \sum_{i,j=1}^{2} (\widehat{a}\widehat{a}_{ij} \frac{\partial \widehat{\widetilde{u}}_p^h}{\partial \widehat{x}_i}, \frac{\partial \widehat{v}_p^h}{\partial \widehat{x}_j})_{m,\widehat{\Omega}},$$

and $\widehat{a}_{ij}$ are the entries of the matrix

$$\widehat{J}_k^h (\widehat{D(T_k^h)^{-1}})(\widehat{D(T_k^h)^{-1}})^t .$$

Here, $\widehat{a}$, $\widehat{a}_{ij}$, $\widehat{\widetilde{u}_p^h}$ and $\widehat{v}_p^h$ denote the restrictions $\widehat{a}_{\Omega_k^h}$, $\widehat{(a_{ij})}_{\Omega_k^h}$, $\widehat{(\widetilde{u}_p^h)}_{\Omega_k^h}$ and $(\widehat{v}_p^h)_{\Omega_k^h}$ respectively.

Let us now derive an estimate of the error $\|u - \widetilde{u}_p^h\|_{1,\Omega}$ for the hp–version under numerical quadrature rules I_m. In fact, $\|u - \widetilde{u}_p^h\|_{1,\Omega}$ depends on two separate terms. The first dependence is on the error $\|u - u_p^h\|_{1,\Omega}$ given in (2.24). Next, the smoothness of a has influence upon the error. We will start with the following Lemma.

LEMMA 3.1. *Let u be the exact solution of (2.6) and u_p^h that of (2.17). Let $\widetilde{u}_p^h$ be an approximate solution of u which satisfies a discrete variational form (3.6). Then there exists a constant C independent of m such that*

$$(3.8) \quad \|u-\widetilde{u}_p^h\|_{1,\Omega}$$

$$\leq C \inf_{v_p^h \in S_{p,0}^h(\Omega)} \{\|u-v_p^h\|_{1,\Omega} + \sup_{w_p^h \in S_{p,0}^h(\Omega)} \frac{|B(u_p^h, w_p^h) - B_{m,\Omega}(v_p^h, w_p^h)|}{\|w_p^h\|_{1,\Omega}}\}.$$

Proof. Let v_p^h be an arbitrary element in $S_{p,0}^h(\Omega)$. Then we have

$$(3.9) \qquad \|u-\widetilde{u}_p^h\|_{1,\Omega} \leq \|u-v_p^h\|_{1,\Omega} + \|v_p^h - \widetilde{u}_p^h\|_{1,\Omega}.$$

From the ellipticity of $B_{m,\Omega}(\cdot,\cdot)$, for a constant $C_1 > 0$

$$\begin{aligned}(3.10) \qquad C_1\|v_p^h - \widetilde{u}_p^h\|_{1,\Omega}^2 &\leq B_{m,\Omega}(v_p^h - \widetilde{u}_p^h, v_p^h - \widetilde{u}_p^h) \\ &= |B_{m,\Omega}(v_p^h, v_p^h - \widetilde{u}_p^h) - (f, v_p^h - \widetilde{u}_p^h)| \\ &= |B_{m,\Omega}(v_p^h, v_p^h - \widetilde{u}_p^h) - B(u_p^h, v_p^h - \widetilde{u}_p^h)|.\end{aligned}$$

Hence, taking the infimum with respect to $v_p^h \in S_{p,0}^h(\Omega)$ we have

$$(3.11) \quad \|u-\widetilde{u}_p^h\|_{1,\Omega}$$

$$\leq C \inf_{v_p^h \in S_{p,0}^h(\Omega)} \{\|u-v_p^h\|_{1,\Omega} + \frac{|B(u_p^h, v_p^h - \widetilde{u}_p^h) - B_{m,\Omega}(v_p^h, v_p^h - \widetilde{u}_p^h)|}{\|v_p^h - \widetilde{u}_p^h\|_{1,\Omega}}\}.$$

The Lemma follows from taking $w_p^h = v_p^h - \widetilde{u}_p^h \in S_{p,0}^h(\Omega)$.

The following Lemma will be used later.

LEMMA 3.2. *Let* $\widehat{u}_p, \widehat{w}_p \in U_p(\widehat{\Omega})$ *and* $\widehat{f} \in L_\infty(\widehat{\Omega})$. *Then, for all* $\widehat{v}_q \in U_q(\widehat{\Omega})$, $\widehat{f}_r \in U_r(\widehat{\Omega})$ *with* $0 < q \leq p$ *and* $r = d(m) - p - q > 0$ *we have*

$$\begin{aligned}(3.12) \qquad & |(\widehat{f}\,\widehat{u}_p, \widehat{w}_p)_{\widehat{\Omega}} - (\widehat{f}\,\widehat{u}_p, \widehat{w}_p)_{m,\widehat{\Omega}}| \\ & \leq C\{\|\widehat{f}_r\|_{0,\infty,\widehat{\Omega}}\|\widehat{u}_p - \widehat{v}_q\|_{0,\widehat{\Omega}} + \|\widehat{f} - \widehat{f}_r\|_{0,\infty,\widehat{\Omega}}\|\widehat{u}_p\|_{0,\widehat{\Omega}}\}\|\widehat{w}_p\|_{0,\widehat{\Omega}},\end{aligned}$$

where C *is independent of* p, q *and* m.

Proof. For any $\widehat{f}_r \in U_r(\widehat{\Omega})$ we have

$$(3.13)\qquad \begin{aligned} &|(\widehat{f}\,\widehat{u}_p, \widehat{w}_p)_{\widehat{\Omega}} - (\widehat{f}\,\widehat{u}_p, \widehat{w}_p)_{m,\widehat{\Omega}}| \\ &\le |(\widehat{f}\,\widehat{u}_p, \widehat{w}_p)_{\widehat{\Omega}} - (\widehat{f}_r\widehat{u}_p, \widehat{w}_p)_{\widehat{\Omega}}| + |(\widehat{f}_r\widehat{u}_p, \widehat{w}_p)_{\widehat{\Omega}} - (\widehat{f}_r\widehat{u}_p, \widehat{w}_p)_{m,\widehat{\Omega}}| \\ &\quad + |(\widehat{f}_r\widehat{u}_p, \widehat{w}_p)_{m,\widehat{\Omega}} - (\widehat{f}\,\widehat{u}_p, \widehat{w}_p)_{m,\widehat{\Omega}}|. \end{aligned}$$

Thank to (K4),

$$(3.14)\qquad (\widehat{f}_r\widehat{v}_q, \widehat{w}_p)_{\widehat{\Omega}} - (\widehat{f}_r\widehat{v}_q, \widehat{w}_p)_{m,\widehat{\Omega}} = 0 \quad \text{for any} \quad \widehat{v}_q \in U_q(\widehat{\Omega}).$$

Hence,

$$(3.15)\qquad \begin{aligned} &|(\widehat{f}_r\widehat{u}_p, \widehat{w}_p)_{\widehat{\Omega}} - (\widehat{f}_r\widehat{u}_p, \widehat{w}_p)_{m,\widehat{\Omega}}| \\ &\le |(\widehat{f}_r\widehat{u}_p, \widehat{w}_p)_{\widehat{\Omega}} - (\widehat{f}_r\widehat{v}_q, \widehat{w}_p)_{\widehat{\Omega}}| + |(\widehat{f}_r\widehat{v}_q, \widehat{w}_p)_{m,\widehat{\Omega}} - (\widehat{f}_r\widehat{u}_p, \widehat{w}_p)_{m,\widehat{\Omega}}|. \end{aligned}$$

By the Schwarz inequality we obtain

$$(3.16)\qquad \begin{aligned} &|(\widehat{f}_r\widehat{u}_p, \widehat{w}_p)_{\widehat{\Omega}} - (\widehat{f}_r\widehat{v}_q, \widehat{w}_p)_{\widehat{\Omega}}| \\ &\le (\widehat{f}_r(\widehat{u}_p - \widehat{v}_q), \widehat{f}_r(\widehat{u}_p - \widehat{v}_q))_{\widehat{\Omega}}^{\frac{1}{2}} (\widehat{w}_p, \widehat{w}_p)_{\widehat{\Omega}}^{\frac{1}{2}} \\ &\le C\,\|\widehat{f}_r\|_{0,\infty,\widehat{\Omega}} \|\widehat{u}_p - \widehat{v}_q\|_{0,\widehat{\Omega}} \|\widehat{w}_p\|_{0,\widehat{\Omega}}\ . \end{aligned}$$

Also, from (K2) we have

$$(3.17)\qquad \begin{aligned} &|(\widehat{f}_r\widehat{v}_q, \widehat{w}_p)_{m,\widehat{\Omega}} - (\widehat{f}_r\widehat{u}_p, \widehat{w}_p)_{m,\widehat{\Omega}}| \\ &\le (\widehat{f}_r(\widehat{u}_p - \widehat{v}_q), \widehat{f}_r(\widehat{u}_p - \widehat{v}_q))_{m,\widehat{\Omega}}^{\frac{1}{2}} (\widehat{w}_p, \widehat{w}_p)_{m,\widehat{\Omega}}^{\frac{1}{2}} \\ &\le C\|\widehat{f}_r\|_{0,\infty,\widehat{\Omega}} (\widehat{u}_p - \widehat{v}_q, \widehat{u}_p - \widehat{v}_q)_{m,\widehat{\Omega}}^{\frac{1}{2}} (\widehat{w}_p, \widehat{w}_p)_{m,\widehat{\Omega}}^{\frac{1}{2}} \\ &\le C\|\widehat{f}_r\|_{0,\infty,\widehat{\Omega}} \|\widehat{u}_p - \widehat{v}_q\|_{0,\widehat{\Omega}} \|\widehat{w}_p\|_{0,\widehat{\Omega}} \end{aligned}$$

Hence, combining (3.16) and (3.17) we estimate

$$(3.18)\qquad \begin{aligned} &|(\widehat{f}_r\widehat{u}_p, \widehat{w}_p)_{\widehat{\Omega}} - (\widehat{f}_r\widehat{u}_p, \widehat{w}_p)_{m,\widehat{\Omega}}| \\ &\le C\,\|\widehat{f}_r\|_{0,\infty,\widehat{\Omega}} \|\widehat{u}_p - \widehat{v}_q\|_{0,\widehat{\Omega}} \|\widehat{w}_p\|_{0,\widehat{\Omega}}\ . \end{aligned}$$

Similarly, since $\widehat{f} \in L_\infty(\widehat{\Omega})$ we obtain

$$
\begin{aligned}
(3.19)\quad & |(\widehat{f}\,\widehat{u}_p,\widehat{w}_p)_{\widehat{\Omega}} - (\widehat{f}_r\widehat{u}_p,\widehat{w}_p)_{\widehat{\Omega}}| \\
& \le ((\widehat{f}-\widehat{f}_r)\widehat{u}_p,(\widehat{f}-\widehat{f}_r)\widehat{u}_p)^{\frac{1}{2}}_{\widehat{\Omega}}\,(\widehat{w}_p,\widehat{w}_p)^{\frac{1}{2}}_{\widehat{\Omega}} \\
& \le C\,\|\widehat{f}-\widehat{f}_r\|_{0,\infty,\widehat{\Omega}}\|\widehat{u}_p\|_{0,\widehat{\Omega}}\|\widehat{w}_p\|_{0,\widehat{\Omega}}\,,
\end{aligned}
$$

and

$$
\begin{aligned}
(3.20)\quad & |(\widehat{f}_r\widehat{u}_p,\widehat{w}_p)_{m,\widehat{\Omega}} - (\widehat{f}\,\widehat{u}_p,\widehat{w}_p)_{m,\widehat{\Omega}}| \\
& \le ((\widehat{f}_r-f)\widehat{u}_p,(\widehat{f}_r-\widehat{f})\widehat{u}_p)^{\frac{1}{2}}_{m,\widehat{\Omega}}(\widehat{w}_p,\widehat{w}_p)^{\frac{1}{2}}_{m,\widehat{\Omega}} \\
& \le C\,\|\widehat{f}_r-\widehat{f}\|_{0,\infty,\widehat{\Omega}}(\widehat{u}_p,\widehat{u}_p)^{\frac{1}{2}}_{m,\widehat{\Omega}}(\widehat{w}_p,\widehat{w}_p)^{\frac{1}{2}}_{m,\widehat{\Omega}} \\
& \le C\,\|\widehat{f}_r-\widehat{f}\|_{0,\infty,\widehat{\Omega}}\|\widehat{u}_p\|_{0,\widehat{\Omega}}\|\widehat{w}_p\|_{0,\widehat{\Omega}}\,.
\end{aligned}
$$

The Lemma follows from (3.18), (3.19), (3.20) and (3.13).

As seen in Lemma 3.1, the last dependence of $\|u-\widetilde{u}^h_p\|_{1,\Omega}$ is on the smoothness of a. In this connection, we let

$$
(3.21)\qquad M_{p,q} = \max_{i,j}\|\widehat{a}_{ij}\|_{p,q,\widehat{\Omega}},
$$

where the subscript q will be omitted when $q=2$. Then, we obtain the following results which give an estimate for the last term of the right side in (3.8).

LEMMA 3.3. *Let $I_m \in G_p$ be a quadrature rule defined on $\widehat{\Omega} \subset R^2$, which satisfies $d(m)-p-1>0$. Let $u \in H^\sigma(\Omega)$, $a \in H^\alpha(\Omega)$ and $\widehat{a}_{ij} \in H^\rho(\widehat{\Omega})$ for $i,j=1,2$, such that $\lambda=\min(\alpha,\rho)\ge 2$. Then, for any $w^h_p \in S^h_{p,0}(\Omega)$ and an approximation u^h_p which satisfies (2.17) we have*

$$
\begin{aligned}
(3.22)\quad & \frac{|\,B(u^h_p,w^h_p)-B_{m,\Omega}(u^h_p,w^h_p)\,|}{\|w^h_p\|_{1,\Omega}} \\
& \le C\,\{q^{-(\sigma-1)}h^{\mu}\|u\|_{\sigma,\Omega} + r^{-(\lambda-1)}h^{(\alpha-1)}\|a\|_{\alpha,\Omega}M_\rho\|u\|_{1,\Omega}\},
\end{aligned}
$$

where $\mu=\min(p,\sigma-1)$ and q is a positive integer such that $0<q\le p$ and $r=d(m)-p-q>0$.

Proof. For arbitrary $w_p^h \in S_{p,0}^h(\Omega)$ we have

$$(3.23)\qquad | B(u_p^h, w_p^h) - B_{m,\Omega}(u_p^h, w_p^h) |$$
$$\leq C \max_{\Omega_k^h \in \mathcal{J}^h} \max_{i,j} \Big| \left(\widehat{a}\,\widehat{a}_{ij} \frac{\partial \widehat{u}_p^h}{\partial \widehat{x}_i}, \frac{\partial \widehat{w}_p^h}{\partial \widehat{x}_j} \right)_{\widehat{\Omega}} - \left(\widehat{a}\,\widehat{a}_{ij} \frac{\partial \widehat{u}_p^h}{\partial \widehat{x}_i}, \frac{\partial \widehat{w}_p^h}{\partial \widehat{x}_j} \right)_{m,\widehat{\Omega}} \Big|.$$

For any $\widehat{a}_{ij}$ $i,j = 1,2$ and $\Omega_k^h \in \mathcal{J}^h$ we let q be any integer such that $0 < q \leq p$ and $r = d(m) - p - q > 0$. Then since $\widehat{a}\,\widehat{a}_{ij} \in L_\infty(\widehat{\Omega})$, due to Lemma 3.2 with $\widehat{v}_q = \frac{\partial}{\partial \widehat{x}_i}(\Pi_q^1 \widehat{u}_p^h)$ and $\widehat{f}_r = \Pi_r^2(\widehat{a}\,\widehat{a}_{ij})$, we have

$$(3.24)\qquad \Big| \left(\widehat{a}\,\widehat{a}_{ij} \frac{\partial \widehat{u}_p^h}{\partial \widehat{x}_i}, \frac{\partial \widehat{w}_p^h}{\partial \widehat{x}_j} \right)_{\widehat{\Omega}} - \left(\widehat{a}\,\widehat{a}_{ij} \frac{\partial \widehat{u}_p^h}{\partial \widehat{x}_i}, \frac{\partial \widehat{w}_p^h}{\partial \widehat{x}_j} \right)_{m,\widehat{\Omega}} \Big|$$
$$\leq C\Big\{ \|\Pi_r^2(\widehat{a}\,\widehat{a}_{ij})\|_{0,\infty,\widehat{\Omega}} \Big\| \frac{\partial \widehat{u}_p^h}{\partial \widehat{x}_i} - \frac{\partial}{\partial \widehat{x}_i}(\Pi_q^1 \widehat{u}_p^h) \Big\|_{0,\widehat{\Omega}}$$
$$+ \|\widehat{a}\,\widehat{a}_{ij} - \Pi_r^2(\widehat{a}\,\widehat{a}_{ij})\|_{0,\infty,\widehat{\Omega}} \Big\| \frac{\partial \widehat{u}_p^h}{\partial \widehat{x}_i} \Big\|_{0,\widehat{\Omega}} \Big\} \Big\| \frac{\partial \widehat{w}_p^h}{\partial \widehat{x}_j} \Big\|_{0,\widehat{\Omega}}.$$

Using Lemma 2.1 and Lemma 2.2 we easily see from the boundedness of Π_q^1 and (2.14) that

$$(3.25)\qquad \Big\| \frac{\partial \widehat{u}_p^h}{\partial \widehat{x}_i} - \frac{\partial}{\partial \widehat{x}_i}(\Pi_q^1 \widehat{u}_p^h) \Big\|_{0,\widehat{\Omega}}$$
$$\leq C \|\widehat{u}_p^h - \Pi_q^1 \widehat{u}_p^h\|_{1,\widehat{\Omega}} \leq C q^{-(\sigma-1)} \|\widehat{u}_p^h\|_{\sigma,\widehat{\Omega}}$$
$$\leq C q^{-(\sigma-1)} \{ \|\widehat{u} - \widehat{u}_p^h\|_{\sigma,\widehat{\Omega}} + \|\widehat{u}\|_{\sigma,\widehat{\Omega}} \}$$
$$\leq C q^{-(\sigma-1)} (h^\mu + h^{(\sigma-1)}) \|u\|_{\sigma,\Omega_k^h}$$
$$\leq C q^{-(\sigma-1)} h^\mu \|u\|_{\sigma,\Omega_k^h},$$

where $\mu = \min(p, \sigma - 1)$.

Also, clearly

$$(3.26)\qquad \Big\| \frac{\partial \widehat{u}_p^h}{\partial \widehat{x}_i} \Big\|_{0,\widehat{\Omega}} \leq C \|\widehat{u}_p^h\|_{1,\widehat{\Omega}} \leq C \|\widehat{u}\|_{1,\widehat{\Omega}} \leq C \|u\|_{1,\Omega_k^h},$$

and

$$(3.27)\qquad \Big\| \frac{\partial \widehat{w}_p^h}{\partial \widehat{x}_j} \Big\|_{0,\widehat{\Omega}} \leq C \|\widehat{w}_p^h\|_{1,\widehat{\Omega}} \leq C \|w_p^h\|_{1,\Omega_k^h}.$$

On the other hand, by an interpolation result (see [9,Theorem 3.2], [7,Theorem 6.2.4]) it follows that for $\widehat{w} \in H^{\eta}(\widehat{\Omega})$ with $\eta \geq 2$,

$$(3.28) \qquad \begin{aligned} &\|\widehat{w} - \Pi_p^2 \widehat{w}\|_{0,\infty,\widehat{\Omega}} \\ &\leq C \|\widehat{w} - \Pi_p^2 \widehat{w}\|_{1+\varepsilon,\widehat{\Omega}}^{\frac{1}{2}} \|\widehat{w} - \Pi_p^2 \widehat{w}\|_{1-\varepsilon,\widehat{\Omega}}^{\frac{1}{2}} \text{ for } 0 < \varepsilon \leq \tfrac{1}{2}. \end{aligned}$$

Also, taking $s = 1 + \varepsilon$ and $s = 1 - \varepsilon$ in (2.21) we have

$$(3.29) \qquad \|\widehat{w} - \Pi_p^2 \widehat{w}\|_{0,\infty,\widehat{\Omega}} \leq C p^{-(\eta-1)} \|\widehat{u}\|_{\eta,\widehat{\Omega}}.$$

Thus, since $\widehat{a}\,\widehat{a}_{ij} \in H^{\lambda}(\widehat{\Omega})$ with $\lambda = \min(\alpha, \rho) \geq 2$ it follows from (3.29) that

$$(3.30) \qquad \begin{aligned} &\|\widehat{a}\,\widehat{a}_{ij} - \Pi_r^2(\widehat{a}\,\widehat{a}_{ij})\|_{0,\infty,\widehat{\Omega}} \\ &\leq C r^{-(\lambda-1)} \|\widehat{a}\|_{\alpha,\widehat{\Omega}} M_\rho \leq C r^{-(\lambda-1)} h^{(\alpha-1)} \|a\|_{\alpha,\Omega_k^h} M_\rho\ . \end{aligned}$$

Moreover, since $\|\Pi_r^2(\widehat{a}\,\widehat{a}_{ij})\|_{0,\infty,\widehat{\Omega}}$ is bounded it follows from (3.25), (3.26), (3.27) and (3.30) that

$$(3.31) \qquad \left| \left(\widehat{a}\,\widehat{a}_{ij} \frac{\partial \widehat{u}_p^h}{\partial \widehat{x}_i}, \frac{\partial \widehat{w}_p^h}{\partial \widehat{x}_j} \right)_{\widehat{\Omega}} - \left(\widehat{a}\,\widehat{a}_{ij} \frac{\partial \widehat{u}_p^h}{\partial \widehat{x}_i}, \frac{\partial \widehat{w}_p^h}{\partial \widehat{x}_j} \right)_{m,\widehat{\Omega}} \right|$$

$$\leq C\{q^{-(\sigma-1)} h^{\mu} \|u\|_{\sigma,\Omega_k^h} + r^{-(\lambda-1)} h^{(\alpha-1)} \|a\|_{\alpha,\Omega_k^h} M_\rho \|u\|_{1,\Omega_k^h}\} \|w_p^h\|_{1,\Omega_k^h},$$

where $\mu = \min(p, \sigma - 1)$.

Consequently, we have

$$(3.32) \qquad \max_{\Omega_i^h \in \mathcal{J}_h} \max_{i,j} \left| \left(\widehat{a}\,\widehat{a}_{ij} \frac{\partial \widehat{u}_p^h}{\partial \widehat{x}_i}, \frac{\partial \widehat{w}_p^h}{\partial \widehat{x}_j} \right)_{\widehat{\Omega}} - \left(\widehat{a}\,\widehat{a}_{ij} \frac{\partial \widehat{u}_p^h}{\partial \widehat{x}_i}, \frac{\partial \widehat{w}_p^h}{\partial \widehat{x}_j} \right)_{m,\widehat{\Omega}} \right|$$

$$\leq C\{q^{-(\sigma-1)} h^{\mu} \|u\|_{\sigma,\Omega} + r^{-(k-1)} h^{(\alpha-1)} \|a\|_{\alpha,\Omega} M_\rho \|u\|_{1,\Omega}\} \|w_p^h\|_{1,\Omega},$$

where $\mu = \min(p, \sigma - 1)$. The Lemma follows from dividing by $\|w_p^h\|_{1,\Omega}$.

By a direct application of Lemma 3.3 and (2.24) to Lemma 3.1 we obtain the following main Theorem which gives an asymptotic, $H^1(\Omega)$-norm error estimate for the rate of convergence under numerical quadrature rules.

THEOREM 3.4. *Let $I_m \in G_p$ be a quadrature rule defined on $\widehat{\Omega} \subset R^2$, which satisfies $d(m) - p - 1 > 0$. We assume that $u \in H^{\sigma}(\Omega), a \in$*

$H^\alpha(\Omega)$ and $\widehat{a}_{ij} \in H^\rho(\widehat{\Omega})$ for $i, j = 1, 2$ such that $\lambda = \min(\alpha, \rho) \geq 2$. Then, for any positive integer q such that $0 < q \leq p$, we have

$$(3.33) \qquad \|u - \widetilde{u}_p^h\|_{1,\Omega} \leq C\{ q^{-(\sigma-1)} h^\mu \|u\|_{\sigma,\Omega} + r^{-(\lambda-1)} h^{(\alpha-1)} \|a\|_{\alpha,\Omega} M_\rho \|u\|_{1,\Omega} \},$$

where $\mu = \min(p, \sigma - 1)$ and $r = d(m) - p - q$.

proof. Taking $v_p^h \in S_{p,0}^h(\Omega)$ with an approximation u_p^h of u which satisfies (2.17), we obtain from Lemma 3.1 that

$$(3.34) \qquad \|u - \widetilde{u}_p^h\|_{1,\Omega} \leq C\{\|u - u_p^h\|_{1,\Omega} + \sup_{w_p^h \in S_{p,0}^h(\Omega)} \frac{|B(u_p^h, w_p^h) - B_{m,\Omega}(u_p^h, w_p^h)|}{\|w_p^h\|_{1,\Omega}} \}.$$

Since $0 < q \leq p$ it follows from (2.24) and Lemma 3.3 that the first term of the right side in (3.34) is dominated by its last term. Hence, the proof is completed by a direct application of Lemma 3.3 to (3.34).

We see from Theorem 3.4 that the rate of convergence is essentially given by

$$(3.35) \qquad O(q^{-(\sigma-1)} h^{\min(p,\sigma-1)} + (d(m) - p - q)^{-(\lambda - \frac{n}{2})} h^{(\alpha-1)}).$$

If m is large enough with $q = p$, then the rate of convergence is asymptotically $O(p^{-(\sigma-1)} h^{min(p,\sigma-1)})$, which coincides with that of (2.24). In the case where a is sufficiently smooth, i.e. α is large enough, even when $d(m) \approx 2p+1$ with $q = p$ the first term in (3.35) may dominate, so that the rate of convergence is asymptotically $O(p^{-(\sigma-1)} h^{min(p,\sigma-1)})$ More precisely, in G-L quadrature rules, using I_m with $(p+1)$-point rules we would obtain an asymptotic rate $O(p^{-(\sigma-1)} h^{\min(p,\sigma-1)})$. But, when a is not smooth enough, the first term $q^{-(\sigma-1)} h^{\min(p,\sigma-1)}$ may be dominated by the other term of (3.35). In this situation, using an overintegration with a sufficiently large m we may reduce the error $\|u - \widetilde{u}_p^h\|_{1,\Omega}$ until the first term dominates again. In practice, when a is not smooth we may increase the value of $d(m)$ with $q \approx p$.

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