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SOME RESULTS ON k^*-PARANORMAL OPERATORS

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1. Introduction

Throughtout this paper, H will always denote a Hilbert space. The $*$-algebra of all bounded linear operators on H is denoted by $L(H)$ and $K(H)$ is the ideal of all compact operators on H. The spectrum, the approximate point spectrum and the essential spectrum of an operator T are denoted by $\sigma(T)$, $\sigma_{ap}(T)$ and $\sigma_e(T)$, respectively. Let $N_T(\mu)$ be the μ-space of T, that is, $N_T(\mu) = \{x \in H : Tx = \mu x\}$. The quotient algebra $L(H)/K(H)$ is called the Calkin algebra. Let $\pi : L(H) \to L(H)/K(H)$ be the natural mapping. Then an operator T is said to be Fredholm if $\pi(T)$ is invertible. The essential spectrum $\sigma_e(T)$ of T is the set of all complex number λ such that $\lambda I - T$ is not a Fredholm operator and is denoted by $\sigma_e(T) = \sigma(\pi(T))$

In [4] and [2], T. Furuta, S.C. Arora and J.K Thukral introduced the concept of paranormal operators and $*$-paranormal operators. $T \in L(H)$ is said to be paranormal if $\|Tx\|^2 \leq \|T^2x\|\|x\|$; $*$-paranormal $\|T^*x\|^2 \leq \|T^2x\|$ for each unit vector x. Motivated by this, we introduce the following : An operator T is said to be a k^*-paranormal operator if $\|T^*x\|^k \leq \|T^kx\|$ for each unit vector x in H. In [7,8], various examples have been constructed to show the proper inclusion relations among the classes of paranormal, $*$-paranormal, and k^*-paranormal operators. It is the aim of this paper to study the properties of the new class of k^*-paranormal operators which generalizes the class of $*$-paranormal operators. Moreover, we show some results on k^*-paranormal operators.

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2. $*$-paranormal operators and k^*-paranormal operators

An operator T is said to be isometric if $\|Tx\| = \|x\|$ for all $x \in H$. It is easy to verify that every isometric operator is hyponormal (i.e $\|T^*x\| \leq \|Tx\|$ for all $x \in H$). An operator T is said to be unitarily equivalent to an operator S if $S = U^*TU$ for an unitary operator U.

In [5], T. Furuta and R. Nakamo have proved the following theorem.

THEOREM A. *A hyponormal operator unitarily equivalent to its adjoint is normal.*

We generalize the above theorem and prove similar results for the class of $*$-paranormal operators and k^*-paranormal operators. In [7, Theoerm 2. 3], we have proved the following theorem.

THEOREM 2.1. *Let T be a $*$-paranormal operator. Then an operator unitarily equivalent to T is $*$-paranormal.*

We generalize Theorem A and have a similar result for k^*-paranormal operator.

THEOREM 2.2. *An operator unitarily equivalent to a k^*-paranormal operator is k^*-paranormal.*

Proof. Suppose $S = U^*TU$, T is k^*-paranormal and U unitary. Now, for each x in H, we have $\|S^*x\|^k = \|U^*T^*Ux\|^k = \|T^*Ux\|^k \leq \|T^kUx\|\|Ux\|^{k-1} = \|US^kx\|\|x\| = \|S^kx\|\|x\|$. Thus S is a k^*-paranormal operator.

THEOREM 2.3. *If T is an isometry and its adjoint is a k^*-paranormal operator, then T is unitary.*

Proof. Since T is an isometry, T is k^*-paranormal. Then we have the following inequality : For any $x \in H$,

$$\begin{aligned}\|x\|^k = \|T^*Tx\|^k \leq \|T^2x\|^k &\leq \|T^{*k}Tx\|\|Tx\|^{k-1} = \|T^{*k-1}x\|\|x\|^{k-1}\\ &\leq \|TT^{*k-2}x\|\|x\|^{k-1} = \|T^{*k-2}x\|\|x\|^{k-1}\\ &\leq \|T^*x\|\|x\|^{k-1} \leq \|Tx\|\|x\|^{k-1} = \|x\|^k\end{aligned}$$

i.e, the equalities $\|Tx\| = \|x\|$ and $\|x\| = \|T^*x\|$ imply $T^*T = I$ and $TT^* = I$, respectively. Therefore T is unitary.

By a similar proof, we can show easily a parallel result for $*$-paranormal operators.

COROLLARY 2.4. *If T is an isometry and its adjoint is a $*$-paranormal operator, then T is unitary.*

In [6], S.M. Patel have proved the following theorem.

THEOREM B. *Let A be a hyponormal operator and let B be $*$-paranormal. If A and B are doubly commutative(i.e. $AB = BA$ and $AB^* = B^*A$), then AB is a $*$-paranormal operator.*

In the following theorem we show that if we replace a hyponormal operator and a $*$-paranormal operator by an isometric operator and a k^*-paranormal operator in Theorem B, then again the condition of commutativity is sufficient to ensure the k^*-paranormality of the product.

THEOREM 2.5. *Let T be a k^*-paranormal operator such that T commutes with an isometric operator S. Then TS is a k^*-paranormal operator.*

Proof. For a unit vector x in H,

$$\begin{aligned}
\|(TS)^*x\|^k &= \|S^*T^*x\|^k \\
&\leq \|ST^*x\|^k \\
&= \|T^*x\|^k \\
&\leq \|T^kx\| \\
&= \|ST^kx\| \\
&= \|S^kT^kx\| \\
&= \|(TS)^kx\|
\end{aligned}$$

Hence, TS is a k^*-paranormal operator.

In [2], S.C.Arora and J.K Thukral showed that power and the inverse(if exists) of $*$-paranormal operator may not be $*$-paranormal. and also this class is not translation invariant.

Product of two commuting $*$-paranormal operators, in general, may not be $*$-paranormal [6, P.94]. The product of two commuting k^*-paranormal operators may not be k^*-paranormal.

In the proof of [1, Theorem 3], T. Ando has showed the following.

LEMMA 2.6. *Let T and S be doublely commuting paranormal operators. If $\|TSx\|\|x\| \geq \|Tx\|\|Sx\|$ (or $\|T^2Sx\|\|x\| \geq \|T^2x\|\|Sx\|$) for all $x \in H$, then TS is a paranormal operator.*

With suitable modification in the inequalities of Lemma 2.6, the following can be showed.

THEOREM 2.7. *Let T and S be doubly commuting $*$-paranormal operators.*

(1) *If $\|T^*Sx\|\|x\| \geq \|T^*x\|\|Sx\|$ for all $x \in H$, then TS is $*$-paranormal.*

(2) *If $\|T^*S^2x\|\|x\| \geq \|T^*x\|\|S^2x\|$ for all $x \in H$, TS is $*$-paranormal.*

Proof. (1) Assume that $\|T^*Sx\|\|x\| \geq \|T^*x\|\|Sx\|$ for all $x \in H$. Since T and S are double commuting $*$-paranormal operators,we have

$$\begin{aligned}\|T^2S^2x\|\|S^2x\|\|Sx\|^2\|T^*x\|\|x\|^2 &\geq \|T^*S^2x\|^2\|Sx\|^2\|T^*x\|\|x\|^2\\ &= \|S^2T^*x\|\|T^*x\|\|S^2T^*x\|\|Sx\|^2\|x\|^2\\ &\geq \|S^*T^*x\|^2\|S^2T^*x\|\|Sx\|^2\|x\|^2\\ &\geq \|S^*T^*x\|^2\|T^*Sx\|\|S^2x\|\|Sx\|\|x\|^2\\ &\geq \|S^*T^*x\|^2\|T^*x\|\|S^2x\|\|Sx\|^2\|x\|,\end{aligned}$$

Hence, $\|(TS)^2x\|\|x\| \geq \|(TS)^*x\|^2$. Thus TS is a $*$-paranormal operator.

(2) By a similar method we have

$$\begin{aligned}\|T^2S^2x\|\|S^2x\|\|S^*x\|\|T^*x\|\|x\| &\geq \|T^*S^2x\|^2\|S^*x\|\|T^*x\|\|x\|\\ &\geq \|S^*T^*x\|^2\|S^*x\|\|x\|\|T^*S^2x\|\\ &\geq \|S^*T^*x\|^2\|T^*x\|\|S^2x\|\|S^*x\|\end{aligned}$$

Hence, $\|(TS)^2x\|\|x\| \geq \|(TS)^*x\|^2$. Thus (TS) is a $*$-paranormal operator.

Modifying the condition (1) in Theorem 2.7, we have a parallel result for a k^*-paranormal operator in the following.

REMARK. *Let T and S be k^*-paranormal operators such that T and S are doubly commutative. If $\|T^*S^kx\|\|x\| \geq \|T^*x\|\|S^kx\|$ for all $x \in H$, where k is a positive integers($k \geq 2$), then TS is a k^*-paranormal operator.*

Proof. Assume that $\|T^*S^kx\|\|x\| \geq \|T^*x\|\|S^kx\|$ for all $x \in H$ and k is a positive integer($k \geq 2$). Since T and S are doubly commuting k^*-paranormal operators, we have

$$\begin{aligned}
\|T^kS^kx\|\|S^kx\|^{k-1}\|T^*x\|^{k-1}\|x\|^k &\geq \|T^*S^kx\|^k\|T^*x\|^{k-1}\|x\|^k \\
&= \|S^kT^*x\|\|S^kT^*x\|^{k-1}\|T^*x\|^{k-1}\|x\|^k \\
&\geq \|S^*T^*x\|^k\|S^kT^*x\|^{k-1}\|x\|^k \\
&= \|S^*T^*x\|^k\|T^*S^kx\|^{k-1}\|x\|^k \\
&\geq \|S^*T^*x\|^k\|T^*x\|^{k-1}\|S^kx\|^{k-1}\|x\|^{k-1}
\end{aligned}$$

Hence, $\|(TS)^kx\|\|x\| \geq \|(TS)^*x\|^k$. Thus TS is a k^*-paranormal operator.

3. Fredholm operators and k^*-paranormal operators

LEMMA 3.1 [3]. *If $T \in L(H)$, the following are equivalent:*

(1) $T \in F_l$

(2) *$ranT$ is closed and $kerT$ is finite dimensional.*

(3) *There is no sequence of unit vectors$\{x_n\}$ such that $lim\|Tx_n\| = 0$ and $x_n \to 0$ weakly.*

(4) *There is no orthonormal sequence $\{e_n\}$ such that $lim\|Te_n\| \to 0$, where F_l, F_r and F denote the left Fredholm, right Fredholm, and Fredholm operators.*

LEMMA 3.2 [3].

(1) *F_l, F_r and F are open.*

(2) *$T \in F_l$ if and only if $T^* \in F_r$*

LEMMA 3.3 [3]. *If $T \in L(H)$, the following are equivalent:*

(1) *T is right invertible*

(2) *T is surjective*

(3) $\inf\{\|T^*x\| : \|x\| = 1\} > 0$

(4) *T^* is left invertible*

(5) *$ranT^*$ is closed and $ker(T^*) = (0)$*

THEOREM 3.4. *Let T be a k^*-paranormal operator.*

(1) *T is invertible if and only if T is right invertible.*

(2) *T is Fredholm if and only if $\pi(T)$ has a right inverse in $L(H)/K(H)$.*

(3) *$\sigma(T) = \sigma_r(T)$ and $\sigma_e(T) = \sigma_{re}(T)$ ($\sigma_r(T)$: right spectrum).*

(4) *$\sigma(T^*) = \sigma_{ap}(T^*)$.*

Proof. (1) Suppose $TB = I$ where $B \in L(H)$. Then $(TB)^* = B^*T^* = I$. Hence $kerT^* = (0)$; and since T is k^*-paranormal, we have $ker(T) = (0)$. Hence T is injective and T is surjective by Lemma 3.3. Therefore T is invertible.

(2) By Lemma 3.1 and Lemma 3.2, $T^* \in F_l$ if and only if $T \in F_r$. Hence $ranT^*$ is closed and $KerT^*$ is finite dimensional, and $ranT$ is closed if and only if $ranT^*$ is closed. Since T is k^*-paranormal, $KerT$ is finite dimensional. Therefore F is a Fredholm operator.

(3) This is immediate from (1) and (2).

(4) By Lemma 3.3 and (3) in Theorem 3.4 we have $\sigma_l(T^*) = \sigma(T^*)$, where $\sigma_l(T^*)$ denote the left spectrum of T^*. And by [4,p 37] we have $\sigma_{ap}(T^*) = \sigma_l(T^*) = \sigma(T^*)$.

THEOREM 3.5. *Let T be a k^*-paranormal operator. Then $\lambda \in \sigma_{ap}(T)$ if and only if there is a $*$-homomorphism $\phi : C^*(T) \to \mathbb{C}$ such that $\phi(T) = \lambda$ where $C^*(T)$ is the C^*-algebra generated by a single operator T.*

Proof. Suppose $\phi : C^*(T) \to \mathbb{C}$ is a $*$-homomorphism such taht $\phi(T) = \lambda$. If $\lambda \notin \sigma_{ap}(T)$, then there is a constant $c > 0$ such that $\|(T - \lambda x\| \geq c\|x\|$ for all $x \in H$. This implies that $T^*T - \lambda T^* - \bar{\lambda}T + \bar{\lambda}\lambda - c^2$

is a positive operator. Hence $0 \leq \phi(T^*T - \lambda T^* - \bar{\lambda}T + \bar{\lambda}\lambda) - c^2 = -c^2$, a contradiction. Hence $\lambda \in \sigma_{ap}(T)$. Conversely, suppose $\lambda \in \sigma_{ap}(T)$. Let $\{x_n\}$ be a sequence of unit vectors in H such that $\|(T-\lambda)x_n\| \to 0$. Let LIM denote a Banach limit and define $\phi : L(H) \to \mathbb{C}$ by $\phi(B) = LIM < Bx_n, x_n >$. If $B \in L(H)$, then $\|B(T-\lambda)x_n\| \to 0$. So $\phi(B(T-\lambda)) = LIM < B(T-\lambda)x_n, x_n >= 0$. Since T is k^*-paranormal, $\|(T-\lambda)^*x_n\| \to 0$. Therefore $\phi(B(T-\lambda)^*) = 0$ for every B in $L(H)$ and $\phi(I) = LIM\|x_n\|^2 = 1$. Therefore if $p((T-\lambda),(T-\lambda)^*)$ is any non-commuting polynomial in $T-\lambda$ and $(T-\lambda)^*$ that has no constant term, $\phi(p((T-\lambda),(T-\lambda)^*)+\alpha) = \alpha$ for all α in $\mathbb{C}$.This implies that ϕ is multiplicative on a dense subalgebra of $C^*(T)$. Hence $\phi|_{C^*(T)}$ is multiplicative and

$$\begin{aligned} 0 = \phi(T-\lambda) &= LIM < (T-\lambda)x_n, x_n > \\ &= LIM < Tx_n, x_n > + LIM < -\lambda x_n, x_n > \\ &= \phi(T) - \lambda \end{aligned}$$

So $\phi(T) = \lambda$ and $\phi(T^*) = LIM < T^*x_n, x_n >= \{LIM < Tx_n, x_n >\}^*$ $= (\phi(T))^*$. Therefore ϕ is a $*$-homomorphism such that $\phi(T) = \lambda$.

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