East Asian Math J 14 (1998), No 1, pp. 21–26

AN EXTENSION OF THE FUGLEDE-PUTNAM THEOREM TO k-QUASIHYPONORMAL OPERATORS

KYO IL SHIN AND HYUNG KOO CHA

ABSTRACT. The Fulgede-Putnam theorem asserts as if A and B are normal operators and X is an operator such that $AX = XB$, then $A^*X = XB^*$. In this paper, we show that if A is k-quasihyponormal and B^* is invertible k-quasihyponormal such that $AX = XB$ for a Hilbert-Schmidt operator X , then $A^*X = XB^*$.

1. Introduction

Let $\mathcal{H}$ be a separable complex Hilbert space and $\mathcal{L}(\mathcal{H})$ the $*$-algebra of all bounded linear operators acting on $\mathcal{H}$. An operator T in $\mathcal{L}(\mathcal{H})$ is called *normal* if $T^*T = TT^*$, *hyponormal* if $T^*T \geq TT^*$, *quasihyponormal* if $T^*(T^*T-TT^*)T \geq 0$ and $k-$*quasihyponormal* for a positive integer k if $T^{*k}(T^*T - TT^*)T^k \geq 0$ which is equivalent to $\|T^{k+1}x\| \geq \|T^*T^kx\|$ for all x in $\mathcal{H}$. It is well known that the following inclusion relations of the classes of nonnormal operators defined above are as follows and they are proper ([5],[6],[9]) ;

Normal $\subsetneqq$ Hyponormal $\subsetneqq$ Quasihyponormal $\subsetneqq$ k-Quasihyponormal.

The classical Fuglede-Putnam theorem is as follows :

THEOREM 1.1. *If A and B are normal operators and if X is an operator such that $AX = XB$, then $A^*X = XB^*$.*

Originally, so called, Fuglede-Putnam theorem has been initiated by Fuglede in [4] under the condition $A = B$ in Theorem 1.1 and one year after Putnam relaxed the condition in [8].

In [2], S. K. Berberian extended the Fuglede-Putnam theorem as follows ·

Received September 29, 1997.

1991 Mathematics Subject Classification. 47B20

Key words and phrases Normal, hyponormal, quasihyponormal, k-quasihyponormal, Hilbert-Schmidt class, trace class.

THEOREM 1.2. *Suppose A, B, X are operators in Hilbert space $\mathcal{H}$, such that $AX = XB$. Assume also that X is an operator of Hilbert-Schmidt class. Then $A^*X = XB^*$ under either of the following hypotheses :*

(1) *A and B^* are hyponormal ;*
(2) *B is invertible and $\|A\|\,\|B^{-1}\| \leq 1$.*

In [5], T. Furuta relaxes the hypotheses on A and B in Theorem 1.2 at the cost of requiring X to be of Hilbert-Schmidt class as follows :

THEOREM 1.3. *Suppose A, B and X are operators on the Hilbert space $\mathcal{H}$ such that $AX = XB$. Assume also that X is an operator of Hilbert-Schmidt class. Then $A^*X = XB^*$ under any one of the following hypotheses :*

(1) *A is k-quasihyponormal and B^* is invertible hyponormal*
(2) *A is quasihyponormal and B^* is invertible hyponormal.*

In this paper, we show that the hyponormality of B^* in Theorem 1.3 can be relaxed by the k-quasihyponormality of B^*.

2. The main theorem

Let T be an operator in $\mathcal{L}(\mathcal{H})$ and suppose that $\{e_n\}$ is an orthonormal basis for $\mathcal{H}$. We define the Hilbert-Schmidt norm of T to be

$$\|T\|_2 = (\sum_{n=1}^{\infty} \|Te_n\|^2)^{\frac{1}{2}}.$$

This definition is independent of the choice of an orthonormal basis ([3],[7]).

If $\|T\|_2 < \infty$, T is said to be a *Hilbert – Schmidt* operator. By $\mathcal{B}_2(\mathcal{H})$ we define the set of all Hilbert-Schmidt operator on $\mathcal{H}$. Let $\mathcal{B}_1(\mathcal{H}) = \{C = AB \mid A, B \in \mathcal{B}_2(\mathcal{H})\}$. Then operators belonging to $\mathcal{B}_1(\mathcal{H})$ are called *trace class* operators. If $\{e_n\}$ is an orthonormal basis for $\mathcal{H}$, we define a linear functional $tr \ : \ \mathcal{B}_1(\mathcal{H}) \to \mathbb{C}$ by

$$tr(C) = \sum_{n=1}^{\infty} < Ce_n, e_n >,$$

then the definition of $tr(C)$ does not depend on the choice of an orthonormal basis and $tr(C)$ is called the trace of C ([3],[7]). The following theorem in [3], [7] is well known :

THEOREM 2.1. *a) The set $\mathcal{B}_2(\mathcal{H})$ is a closed self-adjoint ideal of $\mathcal{L}(\mathcal{H})$;*

b) If $< A\,, B >= \sum_{n=1}^{\infty} < Ae_n, Be_n >= tr(B^*A) = tr(AB^*)$ for A and B in $\mathcal{B}_2(\mathcal{H})$, then $< \,,\, >$ is an inner product on $\mathcal{B}_2(\mathcal{H})$ and $\mathcal{B}_2(\mathcal{H})$ is a Hilbert space with respect to this inner product, where $\{e_n\}$ is any orthonormal basis for H.

THEOREM 2.2. *If $T \in \mathcal{B}(\mathcal{H})$ and $A \in \mathcal{B}_2(\mathcal{H})$, then $\|TA\|_2 \leq \|T\|\,\|A\|_2$, $\|A^*\|_2 = \|A\|$, $\|AT\|_2 \leq \|T\|\|A\|_2$.*

Let A, B in $\mathcal{L}(\mathcal{H})$, from Theorem 2.1, and Theorem 2.2, we can define an operator $\mathcal{J}$ in $\mathcal{L}(\mathcal{B}_2(\mathcal{H}))$ by

$$\mathcal{J}X = AXB,$$

which is due to Berberian in [2]. Evidently, $\|\mathcal{J}\| \leq \|A\|\|B\|$ and its adjoint $\mathcal{J}^*$ is given by

$$\mathcal{J}^*X = A^*XB^*$$

because of

$$\begin{aligned} < \mathcal{J}^*X, Y > &=< X, \mathcal{J}Y >=< X, AYB >= tr((AYB)^*X) \\ &= tr(XB^*Y^*A^*) = tr(A^*XB^*Y^*) =< A^*XB^*, Y > . \end{aligned}$$

If $A \geq 0$ and $B \geq 0$, then $\mathcal{J} \geq 0$ and $\mathcal{J}^{\frac{1}{2}}X = A^{\frac{1}{2}}XB^{\frac{1}{2}}$ since

$$\begin{aligned} < \mathcal{J}X, X > &= tr(AXBX^*) = tr(A^{\frac{1}{2}}XBX^*A^{\frac{1}{2}}) \\ &= tr((A^{\frac{1}{2}}XB^{\frac{1}{2}})(A^{\frac{1}{2}}XB^{\frac{1}{2}})^*) \geq 0. \end{aligned}$$

LEMMA 2.3. *If A and B^* are k-quasihyponormal, then the operator $\mathcal{J}$ in $\mathcal{L}(\mathcal{B}_2(\mathcal{H}))$ defined by $\mathcal{J}X = AXB$ is also k-quasihyponormal.*

Proof. For every $X \in \mathcal{B}_2(\mathcal{H})$, we have

$$\begin{aligned}
&\mathcal{J}^{*k}(\mathcal{J}^*\mathcal{J} - \mathcal{J}\mathcal{J}^*)\mathcal{J}^k X \\
&= (\mathcal{J}^{*k+1}\mathcal{J}^{k+1} - \mathcal{J}^{*k}\mathcal{J}\mathcal{J}^*\mathcal{J}^k)X \\
&= A^{*k+1}A^{k+1}XB^{k+1}B^{*k+1} - A^{*k}AA^*A^kXB^kB^*BB^{*k} \\
&= (A^{*k+1}A^{k+1} - A^{*k}AA^*A^k)XB^{k+1}B^{*k+1} \\
&\quad + A^{*k}AA^*A^kX(B^{k+1}B^{*k+1} - B^kB^*BB^{*k}).
\end{aligned}$$

In this case, since A and B^* are k-quasihyponormal, it follows that

$$\begin{aligned}
&\left\langle (A^{*k+1}A^{k+1} - A^{*k}AA^*A^k)XB^{k+1}B^{*k+1}, X \right\rangle \\
= \;& tr\left((A^{*k+1}A^{k+1} - A^{*k}AA^*A^k)XB^{k+1}B^{*k+1}X^* \right) \\
= \;& tr\Big(((A^{*k+1}A^{k+1} - A^{*k}AA^*A^k)^{\frac{1}{2}}XB^{k+1}) \\
&\qquad ((A^{*k+1}A^{k+1} - A^{*k}AA^*A^k)^{\frac{1}{2}}XB^{k+1})^* \Big) \\
\geq \;& 0
\end{aligned}$$

and

$$\begin{aligned}
&\left\langle (A^{*k}AA^*A^kX(B^{k+1}B^{*k+1} - B^kB^*BB^{*k}), X \right\rangle \\
= \;& \left\langle A^*A^kX(B^{k+1}B^{*k+1} - B^kB^*BB^{*k}), A^*A^kX \right\rangle \\
= \;& tr\left(A^*A^kX(B^{k+1}B^{*k+1} - B^kB^*BB^{*k})X^*A^{*k}A \right) \\
= \;& tr\Big((A^*A^kX(B^{k+1}B^{*k+1} - B^kB^*BB^{*k})^{\frac{1}{2}}) \\
&\qquad (A^*A^kX(B^{k+1}B^{*k+1} - B^kB^*BB^{*k})^{\frac{1}{2}})^* \Big) \\
\geq \;& 0.
\end{aligned}$$

Therefore, $\mathcal{J}$ is also k-quasihyponormal.

LEMMA 2.4. *If B is invertible k-quasihyponormal, then B^{-1} is also k-quasihyponormal.*

Proof. Since B is k-quasihyponormal, $B^{*k}(B^*B-BB^*)B^k = (B^{*k+1}B^{k+1} - B^{*k}BB^*B^k) \geq 0$. Thus, we have

$$B^{-1}B^*BB^{*-1} - I = (B^{*k}B)^{-1}(B^{*k+1}B^{k+1} - B^{*k}BB^*B^k)(B^*B^k)^{-1} \geq 0.$$

Since $A \geq I$ implies $A^{-1} \leq I$, we have $B^*B^{-1}B^{*-1}B \leq I$.

Therefore, we have

$$\begin{aligned}&(B^{*-1})^{k+1}(B^{-1})^{k+1} - (B^{*-1})^kB^{-1}B^{*-1}(B^{-1})^k\\ =&(B^{*-1})^{k+1}(I - B^*B^{-1}B^{*-1}B)(B^{-1})^{k+1} \geq 0,\end{aligned}$$

which completes the proof.

THEOREM 2.5. *If A is k-quasihyponormal and B^* is invertible k-quasihyponormal such that $AX = XB$ for any operator X in $\mathcal{L}(\mathcal{B}_2(\mathcal{H}))$, then $A^*X = XB^*$.*

Proof. Let $\mathfrak{J}$ be the operator on $\mathcal{B}_2(\mathcal{H})$ defined by $\mathfrak{J}X = AXB^{-1}$ Since $(B^*)^{-1} = (B^{-1})^*$ is k-quasihyponormal by Lemma 2.4. $\mathfrak{J}$ is also k-quasihyponormal by Lemma 2.3. The hypothesis $AX = XB$ implies $\mathfrak{J}X = AXB^{-1} = X$ and this relation yields $\|\mathfrak{J}^*X\| = \|\mathfrak{J}^*\mathfrak{J}^kX\| \leq \|\mathfrak{J}^{k+1}X\| = \|X\|$ by k-quasihyponormality of $\mathfrak{J}$. Since $< \mathfrak{J}X\,, X >= \|X\|^2$, we have $\|\mathfrak{J}^*X - X\|^2 \leq \|X\|^2 - 2\|X\|^2 + \|X\|^2 = 0$ and this implies $\mathfrak{J}^*X = X$. Hence, we obtains $A^*X(B^{-1})^* = \mathfrak{J}^*X = X$. Therefore, $A^*X = XB^*$ which is the desired relation.

References

[1] S K. Berberian, *Note on a theorem of Fuglede and Putnam*, Proc Amer Math Soc. **10** (1959), 175–182.

[2] ———, *Extension of a theorem of Fuglede and Putnam*, Proc. Amer Math. Soc. **71** (1978), 113–114

[3] J. B. Conway, *Subnormal Operators, Research Notes in Math, Vol 51*, Pitman Advanced Pub Program, 1981.
[4] B. Fuglede, *A Commutativity theorem for normal operators*, Proc. Nat. Acad. Sci. **36** (1950), 35–40.
[5] T. Furuta, *On relaxation of normality in the Fuglede-Putnam theorem*, Proc. Amer. Math. Soc. **77** (1979), 324–328.
[6] P. R. Halmos, *A Hilbert Space Problem Book*, Springer-Verlag, 1978.
[7] G. J. Murphy, *C^*-algebra and Operator Theory*, Academic Press Inc, 1990.
[8] C. R. Putnam, *On normal operators in Hilbert space*, Amer. J. Math **73** (1951), 357–362.
[9] N. C Shah and I. H Sheth, *Some Results on Quasihyponormal Operators*, J. Indian Math. Soc. **39** (1975).

Department of mathematics
Hanyang University
Seoul 133-791, Korea