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NOTES ON FUZZY SUBGROUPS

Mohammed Al-Ghamdi

Abstract. Obtained the conditions for the product UV of two fuzzy subgroups U and V to be a fuzzy subgroup Moreover, given an example of two fuzzy subgroups U and V which their product UV does not intersect neither U nor V

1. Introduction

Zadeh introduced the concept of fuzzy subset in 1965 [3]. In 1971, Rosenfeld [2] introduced the concept of fuzzy subgroups of an ordinary group. In 1994, Dib [1] introduced a new approach to define fuzzy groups, which considered as a generalization of Rosenfeld approach. Although in [1] Dib generalized the concept of fuzzy subgroups, it is not studied the conditions, which must be satisfied, to be UV form a fuzzy subgroup.

Our interest in this paper is to study the product of two fuzzy subgroups and when their product forms a fuzzy subgroup.

2. Prerequisites

Most contents of this section are in [1], [2].

Definition 1[1]. The fuzzy space is defined by

$$(X, I) = \{(x, I); x \in I\},$$

where (x, I) is called the fuzzy element of (X, I) and the closed unit interval $I = [0, 1]$ contains all possible membership values of x.

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DEFINITION 2[1]. The fuzzy subspace U of the fuzzy space (X, I) is defined by $U = \{(x, u_x); x \in U_0\}$, where $U_0 \subset X$ and $u_x \subset I$ contains at least one element more than 0.

Algebra of fuzzy subspace : Let $U = \{(x, u_x); x \in U_0\}, V = \{(x, u_x); x \in V\}$ be fuzzy subspaces, then

$$U \cup V = \{(x, u_x \cup v_x); x \in U_0 \cup V_0\},$$

$$U \cap V = \{(x, u_x \cap v_x); x \in U_0 \cap V_0\}.$$

Each fuzzy subset A of X defines the fuzzy subspaces

$$H_0(A) = \{(x, \{0, A(x)\}); A(x) \neq 0\}$$

$$\underline{H}(A) = \{(x, [0, A(x)]); A(x) \neq 0\}$$

DEFINITION 3[1]. A fuzzy function from I^X to I^Y is defined as a function $\underline{F} . X \to Y$ characterized by the ordered pair $(F, \{f_x\}_{x \in X})$ (or simply (F, f_x)), where $F : X \to Y$ is a function from X to Y and $f_x; x \in X$ is a family of functions $f_x : I \to I$ satisfying the conditions

(i) f_x is nondecreasing on I,

(ii) $f_x(0) = 0, f_x(1) = 1$,

such that for every fuzzy subset A of $X, \underline{F}(A)$ is a fuzzy subset of Y, defined by (for all $y \in Y$) :

$$\underline{F}(A)y = \begin{cases} \vee_{x \in F^{-1}(y)} f_x(A(x)) & , \text{ if } F^{-1}(y) \neq \phi \\ 0 & , \text{ if } F^{-1}(y) = \phi, \end{cases}$$

The fuzzy function $\underline{F} = (F, f_x)$ defines a function from the fuzzy space (X, I) to the fuzzy space (Y, I), if it has onto comembership functions f_x

$$\underline{F}(x, I) = (F(x), I).$$

DEFINITION 4[1]. The fuzzy Cartesian product $(X, I) \diamond (Y, K)$ of the two fuzzy spaces (X, L) and (Y, K) is the fuzzy space $(X \times Y, L \diamond K)$, where $L \diamond K$ is a vector lattice on $L \times K$ with the partial ordered relation:

(i) $(r_1, r_2) \leq (s_1, s_2)$ if $r_1 \leq s_1$ and $r_2 \leq s_2$ whenever $s_1 \neq 0$ and $s_2 \neq 0$ for all $r_1, s_1 \in L$ and $r_2, s_2 \in K$.

(ii) $(0, 0) = (s_1, s_2)$ if $s_1 = 0$ or $s_2 = 0$ for all $s_1 \in L$ and $s_2 \in K$

DEFINITION 5[1]. The fuzzy binary operation $\underline{F}$ on the fuzzy space (X, I) is a fuzzy function $\underline{F} = (F, f_{xy}) : X \times X \to X$ from the fuzzy Cartesian product $(X \times X, I \diamond I)$ to (X, I) with onto comembership functions f_{xy}, which satisfy that $f_{xy}(r, s) = 0$ iff $r = 0$ or $s = 0$.

Since the fuzzy binary operation $\underline{F} = (F, f_{xy})$ with onto comembership functions, then

$$\underline{F}(\ (x, y), I \diamond I)) = (\ F(x, y), I\ ).$$

If $U = \{(x, u_x); x \in U_0\}$ and $V = \{(x, v_x); x \in V_0\}$ are fuzzy subspaces of (X, I), then

$$\underline{F}\ (x, u_x) \diamond (y, u_y)) = (\ F(x, y), f_{xy}(v_x, v_y)\ )$$

DEFINITION 6[1]. If $\underline{F} = (F, f_{xy})$ is a fuzzy binary operation on the fuzzy space (X, I); then $(\ (X, I); \underline{F}\ )$ is called a fuzzy group if the fuzzy elements of (X, I) satisfy the known conditions of the ordinary group.

The fuzzy subspace $U = \{(x, u_x); x \in U_0\}$ of the fuzzy group $((X, I); \underline{F})$ is called a fuzzy subgroup [1] iff

(i) (U_0, F) is an ordinary group

(ii) $f_{xy}(u_x, u_y) = u_{xFy}$, for all $x, y \in U_0$.

The two fuzzy subspaces U, V are called associative together if $a(bc) = (ab)c$, where a, b, c are orbitrary fuzzy elements choosen from the fuzzy elements of U and V.

3. Fuzzy subgroups

Let $((X, I); \underline{F})$ be a fuzzy group, where $\underline{F} = (F, f_{xy})$ and $(U; \underline{F}), (V; \underline{F})$ be fuzzy subgroups of $((X, I); \underline{F})$ (where $U = \{(x, u_x); x \in U_0\}, V = \{(x, v_x); x \in V_0\}$). About the product $UV = U\underline{F}V$, we have the following

THEOREM 1. *If $(U, \underline{F}), (V, \underline{F})$ are fuzzy subgroups of the fuzzy group $((X, I); \underline{F})$, then $(UV; \underline{F})$ is also a fuzzy subgroup of $((X, I); \underline{F})$ if the following conditions are satisfied:*

(i) $f_{xy}(u_x, v_x) = f_{x'y'}(u_{x'}, u_{y'})$; for all $xFy = x'Fy'$,

(ii) $UV = VU$,

(iii) U and V are associative together.

Proof. Let $U = \{(x, u_x); x \in U_0\}, V = \{(x, v_x); x \in V_0\}$ be fuzzy subspaces and $(U; \underline{F})$,$(V; \underline{F})$ are fuzzy subgroups.

Condition(i) means that $W = U\underline{F}V = UV$ is a fuzzy subspace of (X, I).

Now, we show that W is closed under the fuzzy binary operation $\underline{F}$: for every fuzzy elements

$$(x, w_x)\underline{F}(y, w_y) = (xFy, w_x f_{xy} w_y)$$

But (x, w_x) and (y, w_y) can be written in the form

$$(x, w_x) = (x_1, u_{x_1})\underline{F}(x_2, v_{x_2})$$

$$(y, w_y) = (y_1, u_{u_y1}\underline{F}(y_2, u_{y_2}).$$

Using that U and V are associative together, then

$$\begin{aligned}(x, w_x)\underline{F}(y, w_y) &= ((x_1, u_{x_1})\underline{F}(x_2, v_{x_2}))\underline{F}((y_1, u_{y_1})\underline{F}(y_2, u_{y_2})) \\ &= (x_1, u_{x_1})\underline{F}((x_2, v_{x_2})\underline{F}(y_1, u_{y_1}))\underline{F}(y_2, v_{y_2})\end{aligned}$$

From the condition of the theorem $UV = VU$, then we can write

$$\begin{aligned}(x, w_x)\underline{F}(y, w_y) &= (x_1, u_{x_1})\underline{F}((z_1, u_{z_1})\underline{F}(z_2, v_{z_2}))\underline{F}(y_2, v_{y_2}) \\ &= ((x_1, u_{x_1})\underline{F}((z_1, v_{u_{z_1}}))\underline{F}((z_2, v_{z_2})\underline{F}(y_2, v_{y_2})) \in UV\end{aligned}$$

where $(z_1, u_{z_1}) \in U$ and $(z_2, u_{z_2}) \in V$. Therefore we have

$$(xFy, w_x f_{xy} w_y) = (xFy, w_{xFy})$$

From which we get

(i) $xFy \in U_0V_0 = W_0$

(ii) $w_x f_{xy} w_y = w_{xFy}$

The proof of Theorem-1 is completed if we notice that for every $x \in V_0, y \in V_0$ and $xFy \in U_0V_0$, then

$$(xFy)^{-1} = y^{-1}Fx^{-1} \in V_0U_0 = U_0V_0$$

Therefore, using (i), (U_0V_0, F) is an ordinary subgroup.

The following examples illustrate cases of fuzzy subspaces U, V different from the case of the above theorem.

EXAMPLE 1. Consider the fuzzy group $(X;\underline{F})$ where $X=\{a,b,c\}$, $\underline{F}=(F,f_{xy})$, where F is defined by the following table and the comembership functions f_{xy} are defined as follows

$$f_{xy}(r,s)=r\wedge s,$$

for all (x,y) except for (b,c) and (c,b) it defined by the relation

$$f_{bc}(r,s)=f_{cb}(r,s)=\sqrt{rs}$$

F	a	b	c
a	a	b	c
b	b	b	a
c	c	a	c

Case-1: *An example of fuzzy subgroups* $(U_{r_1};\underline{F}),(V_{r_2};\underline{F})$ *and* $U_{r_1}V_{r_2}$ *is not a fuzzy subspace:* Let $U_{r_1}=\{(a,[0,r_1]),(b,[0,r_1])\}$ and $V_{r_2}=\{(a,[0,r_2]),(c,[0,r_2])\}$. It is clear that $(U_{r_1};\underline{F})$ and $(V_{r_2};\underline{F})$ are fuzzy subgroups of $((X,I);\underline{F})$. But $U_{r_1}V_{r_2}$ is not a fuzzy subspace if $0<r_1<r_2<1$ since

$$r_1 f_{aa} r_2=r_1\wedge r_2\neq r_1 f_{bc} r_2=\sqrt{r_1r_2}\ \text{ if }\ r_1\neq r_2.$$

Case-2: U_rV_r *is a fuzzy subspace and* $(U_rV_r;\underline{F})$ *is a fuzzy subgroup.*

Case-3: *If we define* $U=\{(b,[0,r_1])\}$, $V=\{(c,[0,r_2])\}(r_1>r_2)$, *then* $(U;\underline{F}),(V;\underline{F})$ *are fuzzy subgroups. But* U *is not associative with* V:

$$((b,[0,r_1])\underline{F}(b,[0,r_1]))\underline{F}(c,[0,r_2])=(a,\sqrt{r_1r_2})$$

and

$$(b,[0,r_1])\underline{F}((b,[0,r_1])\underline{F}(c,[0,r_2]))=(a,r_1\wedge\sqrt{r_1r_2})$$

EXAMPLE 2. Consider the fuzzy group $((X,I);\underline{F})$, where $X=\{1,-1\}, I=[0,1]$ and $\underline{F}=(F,f_{xy})$, where F is the usual multiplication on real numbers and $f_{xy}=\sqrt{f(t)f(s)}$, for all $x,y\in\{-1,1\}$ and

$$f(t)=\begin{cases} t, & 0\le t\le 0.2\\ \frac{1-2t}{3}, & 0.2<t\le 0.25\\ -1/2+8/3t, & 0.25<t\le 0.3\\ t, & 0.3<t\le 0.8\\ \frac{13.6-14t}{3}, & 0.8<t\le 0.85\\ -5.1+20/3t, & 0.85<t\le 0.9\\ t, & 0.9<t\le 1\end{cases}$$

It is easy to verify the following statements

(a)-$((X,I);\underline{F})$ is a fuzzy group

(b)-$U=\{(1,[0.2,0.3]\cup\{0\})\}$, $V=\{1,[0.8,0.9]\cup\{0\})\}$ are fuzzy subspaces, satisfying the following interesting properties

(i) $(U,\underline{F})$ and $(V,\underline{F})$ are fuzzy subgroups.

(ii) $U\underline{F}V$ does not contain neither U nor V, i.e.

$$(U\underline{F}V)\cap U=\emptyset \text{ and } (U\underline{F}V)\wedge V=\emptyset$$

(ii) $U\underline{F}V$ is not closed under $\underline{F}$.

References

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Math. Dept. Faculty of Science
King Abdulaziz University
Saudi Arabia