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NEW INEQUALITIES FOR THE MOMENTS OF GUESSING MAPPING

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ABSTRACT. Using some inequalities for real numbers and integrals we print out here some new inequalities for the moments of guessing mapping which complement the recent results of Arikan [1] and Boztas [2].

1. Introduction

J. L. Massey in [4] considered the problem of guessing the value of a realization of a random variable X by asking questions of the form: "Is X equal to x?" until the answer is "Yes".

Let $G(X)$ denote the number of guesses required by a particular guessing strategy when $X = x$.

Massey observed that $E(G(X))$, the average number of guesses, is minimized by a guessing strategy that guesses the possible values of X in decreasing order of probability.

We begin by giving a formal and generalized statement of the above problem by following E. Arikan [1].

Let (X, Y) be a pair of random variables with X taking values in a finite set $\mathcal{X}$ of size u, Y taking values in a countable set $\mathcal{Y}$. Call a fuction $G(X)$ of the random variable X a *guessing function for* X if $G : \mathcal{X} \to \{1, \dots, n\}$ is one-to-one. Call a function $G(X|Y)$ a *guessing function for* X *given* Y if, for any fixed value $Y = y$, $G(X|y)$ is a guessing function for X. $G(X|Y)$ will be thought of as the number of guesses required to determine X where the value of Y is given.

The following inequalities on the moments of $G(X)$ and $G(X|Y)$ were proved by E. Arikow in the recent paper [1].

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THEOREM 1.1. *For an arbitrary guessing function $G(X)$ and $G(X|Y)$ and any $p \geq 0$, we have:*

$$E(G(X)^p) \geq (1+\ln n)^{-p} \left[\sum_{x\in\mathcal{X}} P_X(x)^{\frac{1}{1+p}}\right]^{1+p}$$

and

$$E\left[G(X|Y)^p\right] \geq (1+\ln n)^{-p} \sum_{y\in\mathcal{Y}} \left[\sum_{x\in\mathcal{X}} P_{X,Y}(x,y)^{\frac{1}{1+p}}\right]^{1+p}$$

where $P_{X,Y}, P_X$ are probability distributions of (X,Y) and X, respectively.

Note that, for $p=1$, we get the following estimations on the average number of guesses:

$$E(G(X)) \geq \frac{\left[\sum_{x\in\mathcal{X}} P_X(x)^{\frac{1}{2}}\right]^2}{1+\ln n}$$

and

$$E(G(X|Y)) \geq \frac{\sum_{y\in\mathcal{Y}} \left[\sum_{x\in\mathcal{X}} P_{X,Y}(x,y)^{\frac{1}{2}}\right]^2}{1+\ln n}$$

To simplify the notation further, we assume that the x_i are numbered such that x_k is always the k^{th} guess. This yields

$$E(G^p) = \sum_{k=1}^{n} k^p p_k \quad (p \geq 0).$$

In paper [2] Boztaş proved the following analytic inequality and applied it for the moments of guessing mapping:

THEOREM 1.2. *The relation*

$$\left[\sum_{k=1}^{n} p_k^{1/r}\right]^r \geq \sum_{k=1}^{n} \left(k^r - (k-1)^r\right) p_k \tag{1.1}$$

where $r \geq 1$, holds for any positive integer n, provided that the weights $p_1, \ldots, p_u$ are nonnegative real number satisfying the condition:

$$p_{k+1}^{1/r} \leq \frac{1}{k}\left(p_1^{1/r} + \cdots + p_k^{1/r}\right), \ k = 1, 2, \ldots, n-1. \tag{1.2}$$

If we now consider the guessing problem, we note that (1.1) can be written as [2]:

$$\left[\sum_{k=1}^{n} p_k^{1/(1+p)}\right]^{1+p} \geq E\left(G^{1+p}\right) - E\left[(G-1)^{1+p}\right]$$

for guessing sequences obeying (1.2).

In particular, using the binomial expansion of $(G-1)^{1+p}$ we have the following corollary [2]:

COROLLARY 1.3. *For guessing sequences obeying (1.1) and (1.2) with $r = 1 + m$, the m^{th} guessing moment, $m \geq 1$ being an integer, satisfies:*

$$E\left(G^m\right) \leq \frac{1}{1+m}\left[\sum_{k=1}^{n} p_k^{1/(1_m)}\right]^{1+m} + \frac{1}{1+m}\left\{\binom{m+1}{2} E\left(G^{m-1}\right) - \binom{m+1}{3} E\left(G^{m-1}\right) + \cdots + (-1)^{m+1}\right\}.$$

The following inequalities immediately follow from Corollary 1.3:

$$E(G) \leq \frac{1}{2}\left[\sum_{k=1}^{n} p_k^{1/2}\right]^2 + \frac{1}{2}$$

and

$$E(G^2) \leq \frac{1}{3}\left[\sum_{k=1}^{n} p_k^{1/3}\right]^3 + E[G] - \frac{1}{3}.$$

2. Some new analytic inequalities

We shall start with the following simple integral inequality which is useful in the sequel:

LEMMA 2.1. *Let $f : [0,T] \to \mathbb{R}$ be an integrable mapping on $[0,T]$ with:*

$$(B) \qquad m \le f(x) \le M \quad \text{for all} \quad x \in [0,T], \quad T > 0.$$

Then we have the inequality:

$$\begin{aligned} & m\frac{p}{p+1}T^{p+1} \\ (2.1) \qquad & \le T^p \int_0^T f(u)\,du - \int_0^T u^p f(u)\,du \le M\frac{p}{p+1}T^{p+1} \end{aligned}$$

for all $p > 0$.

Proof. By the condition (B) we get:

$$m\left(T^p - u^p\right) \le \left(T^p - u^p\right) f(u) \le M\left(T^p - u^p\right)$$

for all $u \in [0,T]$ and $p > 1$.

Integrating this inequality on $[0,T]$, we get:

$$\begin{aligned} & m\int_0^T \left(T^p - u^p\right) du \\ & \le T^p \int_0^T f(u)\,du - \int_0^T u^p f(u)\,du \le M\int_0^T \left(T^p - u^p\right) du. \end{aligned}$$

As

$$\int_0^T \left(T^p - u^p\right) du = T^{p+1} - \frac{T^{p+1}}{p+1} = \frac{p}{p+1}T^{p+1}$$

and the inequality (2.1) is proved.

Using this result, we can print out the following discrete inequality which can be applied for the moments of guessing mapping.

THEOREM 2.2. *Let $a_i \in [m, M]$ for all $i = 1, \ldots, n$. Then we have the inequality:*

$$
\begin{aligned}
& m\frac{p}{p+1}n^{p+1} + K \\
(2.2) \qquad & \leq \left[n^p + \frac{(-1)^{p+1}}{p+1}\right]\sum_{i=1}^{n} a_i \leq K + M\frac{p}{p+1}n^{p+1}
\end{aligned}
$$

where

$$
K := \frac{1}{p+1}\left[\binom{p+1}{1}\sum_{i=1}^{n} i^p a_i - \binom{p+1}{2}\sum_{i=1}^{n} i^{p-1} a_i + \ldots + (-1)^{p+1}\binom{p+1}{1}\sum_{i=1}^{n} i\, a_i\right]
$$

and $p \in \mathbb{N}, p \geq 1$.

Proof. Consider the mapping $f : [0, n] \to \mathbb{R}$ given by

$$
f(u) := \begin{cases} a_1, & u \in [0,1) \\ a_2, & u \in [1,2) \\ \cdots\cdots\cdots & \\ a_n, & u \in [n-1, n] \end{cases}
$$

This mapping is integrable on $[0, n]$ and, of course, $m \leq f(x) \leq M$ for all $x \in [a, b]$.

We have

$$
\int_0^n f(u)\,du = \sum_{i=0}^{n-1}\int_i^{i+1} f(u)\,du = \sum_{i=0}^{n-1} a_{i+1} = \sum_{i=1}^{n} a_i
$$

and

$$
\begin{aligned}
I_p := \int_0^n u^p f(u)\,du &= \sum_{i=0}^{n-1}\int_i^{i+1} u^p f(u)\,du = \sum_{i=0}^{n-1}\frac{(i+1)^{p+1} - i^{p+1}}{p+1}a_{i+1} \\
&= \frac{1}{p+1}\sum_{i=1}^{n}\left[i^{p+1} - (i-1)^{p+1}\right] a_i.
\end{aligned}
$$

But

$$
\begin{aligned}
& i^{p+1}-(i-1)^{p+1} \\
& =\binom{p+1}{1} i^p-\binom{p+1}{2} i^{p-1}+\cdots+(-1)^{p+1}\binom{p+1}{1} i+(-1)^{p+2}
\end{aligned}
$$

and thus

$$
\begin{aligned}
I_p=\frac{1}{p+1} & {\left[\binom{p+1}{1} \sum_{i=1}^n i^p a_i-\binom{p+1}{2} \sum_{i=1}^n i^{p-1} a_i+\ldots\right.} \\
& \left.+(-1)^{p+1}\binom{p+1}{1} \sum_{i=1}^n i\, a_i+(-1)^{p+2} \sum_{i=1}^n a_i\right] .
\end{aligned}
$$

Now using Lemma 2.1, we deduce:

$$
\begin{aligned}
& m \frac{p}{p+1} u^{p+1} \\
& \leq u^p \sum_{i=1}^n a_i-\frac{1}{p+1}\left[\binom{p+1}{1} \sum_{i=1}^n i^p a_i-\binom{p+1}{2} \sum_{i=1}^n i^{p-1} a_i+\cdots\right. \\
& \left.\quad+(-1)^{p+1}\binom{p+1}{1} \sum_{i=1}^n i\, a_i+(-1)^{p+2} \sum_{i=1}^n a_i\right] \\
& \leq M \frac{p}{p+1} u^{p+1}
\end{aligned}
$$

which is equivalent to the desired inequality (2.2).

The following result is well known in the literature as the integral Grüss' inequality [5]:

LEMMA 2.3. *Let $h, g : [a, b] \to \mathbb{R}$ be two integrable functions so that*

$$m_1 \leq g(x) \leq M_1, \quad m_2 \leq h(x) \leq M_2 \text{ for all } x \in [a, b].$$

Then we have the estimation:

$$
\begin{aligned}
& \left|\frac{1}{b-a} \int_a^b g(x) h(x)\, d x-\frac{1}{b-a} \int_a^b g(x)\, d x \frac{1}{b-a} \int_a^b h(x)\, d x\right| \\
& \leq \frac{1}{4}\left(M_1-m_1\right)\left(M_2-m_2\right)
\end{aligned} \tag{2.3}
$$

and the constant $\frac{1}{4}$ *is the best possible one.*

The following discrete version of Grüss' inequality is important by its applications for the moments of guessing mapping.

THEOREM 2.4. *Let* $a_i, b_i \in \mathbb{R}$ $(i = \overline{1,n})$ *be so that*

$$a \le a_i \le A, \quad b \le b_i \le B \quad \text{for all} \quad i = \overline{1,n}.$$

Then we have the inequality:

$$\left| \frac{1}{n} \sum_{i=1}^{n} a_i b_i - \frac{1}{n} \sum_{i=1}^{n} a_i \cdot \frac{1}{n} \sum_{i=1}^{n} b_i \right| \le \frac{1}{4}(A-a)(B-b). \tag{2.4}$$

Proof. We choose in Grüss' integral inequality

$$g(x) := \begin{cases} a_1, & x \in [0,1) \\ a_2, & x \in [1,2) \\ \dots\dots\dots & \dots\dots \\ a_n, & x \in [n-1,n] \end{cases}$$

and

$$h(x) := \begin{cases} b_1, & x \in [0,1) \\ b_2, & x \in [1,2) \\ \dots\dots\dots\dots\dots \\ b_n, & x \in [n-1,n] \end{cases}$$

Then $m_1 = a, M_1 = A, m_2 = b$ and $M_2 = B$ and

$$\int_0^n g(x)dx = \sum_{i=1}^{n} a_i, \ \int_0^n h(x)dx = \sum_{i=1}^{n} b_i \text{ and } \int_0^n g(x)h(x)dx = \sum_{i=1}^{n} a_i b_i$$

and the theorem is proved.

The following lemma contains an integral inequality which is interesting in itself too

LEMMA 2.5. *Let $f : [a,b] \to \mathbb{R}$ be an integrable mapping. Then we have the inequality:*

$$\left|\int_a^b (x-a)^p f(x)\,dx - \frac{(b-a)^{p+1}}{p+1}\int_a^b f(x)\,dx\right| \leq \frac{(b-a)^{p+1}}{4}(M-m) \tag{2.5}$$

where

$$M := \sup_{x\in[a,b]} f(x) < \infty \qquad m := \inf_{x\in[a,b]} f(x) > -\infty$$

and $p \geq 0$.

Proof. Follows by Grüss' integral inequality for $g(x) := (x-a)^p$ and $h(x) := f(x), x \in [a,b]$.

We can apply this lemma to prove a discrete inequality which is important by its applications for the moments of guessing mapping.

THEOREM 2.6. *Let $a_i \in [m,M]$ for all $i = 1,\dots,n$. Then we have the inequality:*

$$\left|\binom{p+1}{1}\sum_{i=1}^n i^p a_i - \binom{p+1}{2}\sum_{i=1}^n i^{p-1}a_i + \cdots + (-1)^{p+1}\binom{p+1}{1}\sum_{i=1}^n i\,a_i - \left(n^{p+1}+(-1)^{p+1}\right)\sum_{i=1}^n a_i\right| \leq \frac{(p+1)n^{p+1}}{4}(M-m) \tag{2.6}$$

for all $p \in \mathbb{N}$, $p \geq 1$.

Proof. Let choose in Lemma 2.5, $a=0,\ b=n,\ f(x) = a_{i+1},\ x \in [i,i+1),\ i = 0,\dots,n-1$. Then we have:

$$\int_0^n f(x)\,dx = \sum_{i=1}^n a_i$$

and

$$\int_0^n x^p f(x)\,dx = \frac{1}{p+1}\left[\binom{p+1}{1}\sum_{i=1}^n i^p a_i - \binom{p+1}{2}\sum_{i=1}^n i^{p-1}a_i + \right.$$
$$\left. \cdots + (-1)^{p+1}\binom{p+1}{1}\sum_{i=1}^n i\,a_i + (-1)^{p+2}\sum_{i=1}^n a_i\right].$$

Using the inequality (2.5) we get:

$$\left|\frac{1}{p+1}\left[\binom{p+1}{1}\sum_{i=1}^n i^p a_i - \binom{p+1}{2}\sum_{i=1}^n i^{p-1}a_i + \dots \right.\right.$$
$$\left.\left. + (-1)^{p+1}\binom{p+1}{1}\sum_{i=1}^n i\,a_i + (-1)^{p+2}\sum_{i=1}^n a_i\right] - \frac{n^{p+1}}{p+1}\sum_{i=1}^n a_i\right|$$
$$\leq \frac{n^{p+1}}{4}(M-m)$$

and the inequality (2.6) is obtained.

In paper [3] Dragomir and Ionescu have proved between other the following counterpart of Jensen's inequality for differentiable mappings of a real variable:

THEOREM 2.7. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex differentiable mapping on the interior of I and $p_i \geq 0, x_i \in I$ with $P_n := \sum_{i=1}^n p_i > 0$. Then we have the following counterpart of Jensen's discrete inequality:*

$$(2.7)\quad \begin{aligned} 0 &\leq \frac{1}{P_n}\sum_{i=1}^n p_i f(x_i) - f\left(\frac{1}{P_n}\sum_{i=1}^n p_i x_i\right) \\ &\leq \frac{1}{P_n}\sum_{i=1}^n p_i x_i f'(x_i) - \frac{1}{P_n}\sum_{i=1}^n p_i x_i \frac{1}{P_n}\sum_{i=1}^n p_i f'(x_i). \end{aligned}$$

Proof. For the sake of completeness, we shall give here a short proof. By the convexity of f in I we have that:

$$f(x) - f(y) \geq f'(y)(x-y)$$

for all x, y in the interior of I. Choosing $x = \frac{1}{P_n}\sum_{i=1}^{n} p_i x_i$ and $y = x_j$ we get:

$$f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right) - f(x_j) \geq f'(x_j)\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i - x_j\right).$$

If we multiply by p_j and summing over j to 1 at n we derive:

$$P_n f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right) - \sum_{j=1}^{n} p_j f(x_j)$$
$$\geq \frac{1}{P_n}\sum_{i=1}^{n} p_i x_i \cdot \sum_{j=1}^{n} p_j f'(x_j) - \sum_{j=1}^{n} p_j x_j \cdot f'(x_j)$$

which is obviously equivalent to (2.7).

COROLLARY 2.8. *Let $a_i \geq 0\,(i = \overline{1,n})$. Assume $A_n := \sum_{i=1}^{n} a_i$.*

1 . If $p \geq 1$, then we have the inequality:

$$0 \leq \frac{1}{A_n}\sum_{i=1}^{n} i^p a_i - \left(\frac{1}{A_n}\sum_{i=1}^{n} i\, a_i\right)^p$$
$$\leq \frac{p}{A_n}\sum_{i=1}^{n} i^p a_i - \frac{1}{A_u}\sum_{i=1}^{n} i^{p-1} a_i$$

2 . If $0 \leq p < 1$, then we have the inequality:

$$0 \leq \left(\frac{1}{A_n}\sum_{i=1}^{n} i\, a_i\right)^p - \frac{1}{A_n}\sum_{i=1}^{n} i^p a_i$$
$$\leq \frac{p}{A_n}\sum_{i=1}^{n} i\, a_i \cdot \frac{1}{A_n}\sum_{i=1}^{n} i^{p-1} a_i - \frac{p}{A_n}\sum_{i=1}^{n} i^p a_i.$$

3. Some inequalities for moments of guessing mapping

The following estimation result for the p-moment of the guessing mappings holds.

THEOREM 3.1. *Let X be a random variable having the probability distribution $p = (p_i), i = \overline{1,n}$. Then we have the inequality:*

$$\left|E\left(G(X)^p\right) - \frac{1}{n}\sum_{i=1}^{n} i^p\right| \leq \frac{n(n^p - 1)}{4}(P_M - P_m) \tag{3.1}$$

where

$$P_M := \max\{p_i | i = \overline{1,n}\} \quad \text{and } P_m := \min\{p_i | i = \overline{1,n}\}$$

and $p > 0$.

Proof. We shall apply Theorem 2.4 for $a_i = i^p$ and $b_i = p_i (i = \overline{1,n})$ to get

$$\left|\frac{1}{n}\sum_{i=1}^{n} i^p p_i - \frac{1}{n}\sum_{i=1}^{n} i^p \cdot \frac{1}{n}\sum_{i=1}^{n} p_i\right| \leq \frac{(n^p - 1^p)(P_M - P_m)}{4}$$

which is equivalent to (3.1).

COROLLARY 3.2. *If we assume that for a given $\varepsilon > 0$ and $n \geq 1$, we have*

$$0 \leq P_M - P_m < \frac{4\varepsilon}{n(n^p - 1)}$$

then

$$\left|E(G(X)^p) - \frac{1}{n}\sum_{i=1}^{n} i^p\right| < \varepsilon.$$

REMARK 3.3. If we put in (3.1) $p = 1$, we get:

$$\left|E(G(X)) - \frac{n+1}{2}\right| \le \frac{n(n-1)}{4}(P_M - P_m).$$

If we choose in (3.1) $p = 2$, we get

$$\left|E(G(X)^2) - \frac{(n+1)(2n+1)}{6}\right| \le \frac{n(n^2-1)}{4}(P_M - P_m)$$

and, finally, for $p = 3$, we obtain

$$\left|E(G(X)^3) - \frac{n(n+1)^2}{4}\right| \le \frac{n(n^3-1)}{4}(P_M - P_m).$$

The following theorem also holds.

THEOREM 3.4. *With the assumptions of Theorem 3.1, we have the inequality:*

$$\begin{aligned}&\left|\binom{p+1}{1}E(G(X)^p) - \binom{p+1}{2}E(G(X)^{p-1}) + \dots\right.\\ &\qquad \left. + (-1)^{p+1}\binom{p+1}{1}E(G(X)) + (-1)^{p+2} - n^{p+1}\right| \\ &\le \frac{(p+1)n^{p+1}}{4}(P_M - P_m)\end{aligned} \tag{3.2}$$

provided that $p \in \mathbb{N}$, $p \ge 1$.

Proof. Follows by Theorem 2.6 choosing $a_i = p_i$ and taking into account that $\sum_{i=1}^{n} p_i = 1$.

COROLLARY 3.5. *If we assume that for a given* $\varepsilon > 0$ *and* $n \ge 1$, *we have:*

$$0 \le P_M - P_m < \frac{4\varepsilon}{(p+1)n^{p+1}}$$

then

$$\begin{aligned}&\left|\binom{p+1}{1}E(G(X)^p) - \binom{p+1}{2}E\left(G(X)^{p-1}\right) + \dots\right.\\ &\qquad \left. + (-1)^{p+1}\binom{p+1}{1}E(G(X)) + (-1)^{p+2} - n^{p+1}\right| < \varepsilon.\end{aligned}$$

REMARK 3.6. If in (3.2) we put $p = 1$, we get:

$$\left|E(G(X)) - \frac{n^2+1}{2}\right| \le \frac{n^2}{4}(P_M - P_m)$$

and if we choose $p = 2$, we get.

$$\left|E(G(X)^2) - E(G(X)) - \frac{n^3-1}{3}\right| \le \frac{n^3}{4}(P_M - P_m).$$

Finally, the following theorem also holds.

THEOREM 3.7. *With the assumptions of Theorem 3.4, we have the inequality*

$$\begin{aligned} & P_m \frac{p}{p+1} n^{p+1} \\ (3.3) \quad & \le n^p - \frac{1}{p+1}\left[\binom{p+1}{1} E(G(X)^p) + \dots \right. \\ & \left. + (-1)^{p+1}\binom{p+1}{1} E(G(X)) + (-1)^{p+2}\right] P_M \frac{p}{p+1} n^{p+1} \end{aligned}$$

where $p \in \mathbb{N}$ *and* $p \ge 1$.

Proof. The argument follows by Theorem 2.2 choosing $a_i = p_i$, and taking into account that:

$$\sum_{i=1}^{n} p_i = 1 \quad \text{and } m = P_m,\ M = P_M.$$

We shall omit the details.

COROLLARY 3.8. *With the above assumptions we have:*

$$\begin{aligned} & \left| n^p - \frac{1}{p+1}\left[\binom{p+1}{1} E(G(X))^p) + \dots \right.\right. \\ & \left.\left. + (-1)^{p+1}\binom{p+1}{1} E(G(X)) + (-1)^{p+2}\right] - \frac{p}{p+1}\cdot\frac{P_m + P_M}{2} n^{p+1} \right| \\ & \le \frac{p}{p+1}\cdot\frac{P_M - P_m}{2}\cdot n^{p+1}. \end{aligned}$$

Using Corollary 2.8, we can state the following result for the moments of guessing mapping:

THEOREM 3.9. *Let X be a random variable and $G(X)$ an arbitrary guessing function. Then*

1 . *If $p \geq 1$, then we have the inequality:*

$$0 \leq E(G(X)^p) - [E(G(X))]^p \leq pE(G(X)^p) - E(G(X)^{p-1}).$$

2 . *If $p \in (0,1)$, then we have the reverse inequality, i.e.,*

$$0 \leq [E(G(X))]^p - E(G(X)^p) \leq p[E(G(X))E(G(X)^{p-1}) - E(G(X)^p)].$$

Proof. Follows by Corollary 2.8 choosing $a_i = p_i, i = \overline{1,n}$ and taking into account that $\sum_{i=1}^n p_i = 1$.

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