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SURFACES SATISFYING $\Delta x = Ax$ IN S^4

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1. Introduction

Let M^n be an n-dimensional connected submanifold of the m-dimensional Euclidean space E^m, equipped with the induced metric Denote by Δ the Laplace operator of M^n and by x the position vector. Then the well-known Takahashi's theorem says that M^n satisfies $\Delta x = \lambda x$ for a constant λ if and only if M^n is minimal in $E^m(\lambda = 0)$ or is minimal in a hypersphere of $E^m(\lambda \neq 0)$[6]. In [2], F. Dillen, J. Pas and L. Verstraellen generalized Takahashi's condition and posed to study submanifolds for which

$$\Delta x = Ax + b, \tag{1.1}$$

where A is an $m \times m$ matrix and b is a constant vector in E^m. Hypersurfaces satisfying (1.1) in space forms are completely classified[1,3,4]. In [5] the first author classified Euclidean compact submanifolds of codimension 2 with constant mean curvature satisfying (1.1) and in [4] he proved that if an n-dimensional submanifold M^n of the $(n+2)$-dimensional sphere S^{n+2} satisfy $\Delta x = Ax$ and M^n is fully contained in E^{n+3}, then A must be symmetric. In this paper we study sufaces in S^4 satisfying $\Delta x = Ax$ and obtain the following classification theorem:

Theorem *If M^2 is a surface with constant mean curvature in S^4 and satisfies $\Delta x = Ax$ for a 5×5 matrix A, then M^2 is one of the followings:*

(1) *an open part of 2-sphere,*

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(2) *a minimal surface of a hypersphere in* S^4,
(3) *an open part of a product* $S^1(r_1) \times S^1(r_2)$ *such that* $0 < r_1^2 + r_2^2 \leq 1$, $r_1 \neq r_2$,
(4) *a minimal surface of* S^4.

2. Preliminaries

Let M^2 be a surface in the unit hypersphere S^4 centered at origin in E^5. We will use the same notation $\langle\, , \rangle$ for the Euclidean metric on E^5 and the induced metric on M^2. Denote by ∇ and ∇' the Levi-Civita connections on M^2 and E^5, respectively. Let e_1, e_2 be an orthonormal local tangent frame on M^2. Then the Laplacian Δ is given by

$$\Delta = \sum_{i=1}^{2}(e_i e_i - \nabla_{e_i} e_i).$$

Let H denote the mean curvature vector of M^2. Then we have

$$\Delta x = H,$$

where x denotes the position vector of M^2. Acting Δ to $\langle x, x\rangle = 1$, we obtain

(2-1) $$\langle \Delta x, x\rangle = -2.$$

Let e_3, e_4, e_5 be an orthonormal normal frame on M^2 such that $e_3 = x$. From now on, the indices i, j, k run over the range $\{1, 2\}$ and the indices r, s over $\{3, 4, 5\}$ unless stated otherwise. Denote by ω_B^A, $A, B = 1, 2, \cdots, 5$, the connection forms. Then we have

(2-2)
$$\nabla'_{e_i} e_j = \nabla_{e_i} e_j + h(e_i, e_j),\ \nabla_{e_i} e_j = \sum_k \omega_j^k(e_i) e_k,\ h(e_i, e_j) = \sum_r h_{ij}^r e_r,$$

(2-3)
$$\nabla'_{e_i} e_r = \sum_s \omega_r^s(e_i) e_s - \sum_j h_{ij}^r e_j,$$

where h is the second fundamental form and h_{ij}^r are the coefficients of the second fundamental form h. Note that $h_{ij}^3 = -\delta_{ij}$ and $\omega_r^3(e_i) = 0$. Now assume that M^2 satisfy

(2-4) $$\Delta x = Dx,$$

where $D = \mathrm{diag}[\lambda_1, \cdots, \lambda_5]$ is an 5×5 diagonal matrix and the mean curvature function $|H|$ is constant. Then we get the following four equations:

(2-5) $$\langle x, x\rangle = \sum_{i=1}^{5} x_i^2 = 1,$$

(2-6) $$\langle Dx, x\rangle = \sum_{i=1}^{5} \lambda_i x_i^2 = -2,$$

(2-7) $$\langle D^2x, x\rangle = \sum_{i=1}^{5} \lambda_i^2 x_i^2 = |H|^2,$$

(2-8) $$\langle D^3x, x\rangle = \sum_{i=1}^{5} \lambda_i^3 x_i^3 = -\sum_{i=1}^{2}\langle De_i, e_i\rangle,$$

where x_i are coordinate functions of M^2. Equation (2.6) follows from (2.1), (2.4) and equation (2.8) can be obtained by acting Δ to $\langle D^2x, x\rangle = |H|^2$. Let

(2-9) $$\langle D^k x, x\rangle = d_k$$

for every nonnegative integer k. Note that $d_0 = 1$, $d_1 = -2$ and $d_2 = |H|^2$. If the normal vectors x, Dx and D^2x are linearly independent, then these span the normal space of M^2. Moreover we have the following lemma.

Lemma 1. *If x, Dx, D^2x are locally linearly independent, then M^2 is an open part of a product $S^1(r_1) \times S^1(r_2)$, where $r_1^2 + r_2^2 < 1$ and $r_1 \neq r_2$.*

proof. Assume that x, Dx, D^2x are linearly independent on an open subset U of M^2. Then by (2.5), (2.6), (2.7) and (2.9) the following three vectors form an orthonormal normal frame of U.

$$\text{(2-10)} \qquad \xi_1 = x,\ \xi_2 = \frac{Dx - d_1x}{\sqrt{d_2 - d_1^2}},\ \xi_3 = \frac{D^2x - c(Dx - d_1x) - d_2x}{\sqrt{d_4 - d^2 - c^2(d_2 - d_1^2)}},$$

where $c = \frac{d_3 - d_1d_2}{d_2 - d_1^2}$. From (2.8) it follows that

$$\text{(2-11)} \qquad -\langle D^3x, x\rangle + \langle D^2\xi_1, \xi_1\rangle + \langle D^2\xi_2, \xi_2\rangle + \langle D^2\xi_3, \xi_3\rangle = \text{tr } D^2.$$

Substituting (2.10) into (2.11), equation (2.11) becomes
(2-12)

$$\begin{aligned}
&(\sum_{i=1}^5 \lambda_i^3x_i^2)^3 + 2d_2(1-d_1)(\sum_{i=1}^5 \lambda_i^3x_i^2)^2 + (d_1^2 - d_2)(\sum_{i=1}^5 \lambda_i^3x_i^2)(\sum_{i=1}^5 \lambda_i^4x_i^2)\\
&- 2(\sum_{i=1}^5 \lambda_i^3x_i^2)(\sum_{i=1}^5 \lambda_i^5x_i^2) + (\sum_{i=1}^5 \lambda_i^4x_i^2)^2 + (\text{tr}D^2 - d_2 + d_2^3)(\sum_{i=1}^5 \lambda_i^3x_i^2)\\
&+ \{(d_1^2 - d_2)\text{tr}D^2 - 2d_2^2\}(\sum_{i=1}^5 \lambda_i^4x_i^2) + 2d_1d_2(\sum_{i=1}^5 \lambda_i^5x_i^2)\\
&+ (d_2 - d_1^2)(\sum_{i=1}^5 \lambda_i^6x_i^2) + d_2(d_2^2 - d_2d_1^2 - d_1)\text{tr}D^2 + d_2^2d_1(1 + d_2d_1) = 0.
\end{aligned}$$

Without loss of generality, U can be locally described as the set of points

$$(x_1, x_2, f(x_1,x_2), g(x_1,x_2), h(x_1,x_2)),$$

where f, g, h are smooth functions defined on an open subset of E^2. If any two of $\lambda_3,\ \lambda_4,\ \lambda_5$ are equal, then (2.5), (2.6) and (2.7) imply that Dx and D^2x are linearly dependent. Thus we may assume that $\lambda_3, \lambda_4, \lambda_5$ are mutually different. From (2.5), (2.6) and (2.7) we find

$$\begin{aligned}
x_3^2 &= \frac{\sum_{i=1}^2(\lambda_i - \lambda_4)(\lambda_i - \lambda_5)x_i^2}{(\lambda_3 - \lambda_4)(\lambda_5 - \lambda_3)} + \frac{d_1(\lambda_4 + \lambda_5) - \lambda_4\lambda_5 - d_2}{(\lambda_3 - \lambda_4)(\lambda_5 - \lambda_3)},\\
\text{(2-13)} \quad x_4^2 &= \frac{\sum_{i=1}^2(\lambda_i - \lambda_3)(\lambda_i - \lambda_5)x_i^2}{(\lambda_3 - \lambda_4)(\lambda_4 - \lambda_5)} + \frac{d_1(\lambda_3 + \lambda_5) - \lambda_3\lambda_5 - d_2}{(\lambda_3 - \lambda_4)(\lambda_4 - \lambda_5)},\\
x_5^2 &= \frac{\sum_{i=1}^2(\lambda_i - \lambda_3)(\lambda_i - \lambda_4)x_i^2}{(\lambda_4 - \lambda_5)(\lambda_5 - \lambda_3)} + \frac{d_1(\lambda_3 + \lambda_4) - \lambda_3\lambda_4 - d_2}{(\lambda_4 - \lambda_5)(\lambda_5 - \lambda_3)}.
\end{aligned}$$

Substituting (2.13) into (2.12), we get a polynomial with respect to x_1, x_2, which vanishes identically on an open subset of E^2. However, the coefficients of x_i^6, $i = 1, 2$ is equal to $(\lambda_i - \lambda_3)^3(\lambda_i - \lambda_4)^3(\lambda_i - \lambda_5)^3$. Hence, we have $(\lambda_i - \lambda_3)(\lambda_i - \lambda_4)(\lambda_i - \lambda_5) = 0$. Thus the following two cases are possible

$$\textbf{case 1} \qquad \lambda_1 = \lambda_2 = \lambda_3,$$
$$\textbf{case 2} \qquad \lambda_1 = \lambda_3, \lambda_2 = \lambda_4,$$

If case 1 holds, then x_4 and x_5 are constants. This imply D^2x and Dx are linearly dependent. Thus this case cannot happen. Consider case 2. Then we have $x_1^2+x_3^2$=constant , $x_2^2+x_4^2$=constant and x_5 =nonzero constant. Hence M^2 is an open part of $S^1(r_1) \times S^1(r_2)$, where r_1, r_2 are positive reals such that $r_1^2 + r_2^2 < 1$.

LEMMA 2 *If M^2 is not minimal in S^4 and $D = diag[\lambda_1, \lambda_1, \lambda_1, \lambda_2, \lambda_2]$ with $\lambda_1 \neq \lambda_2$, then M^2 is an open part of a 2-sphere($\lambda_2 = 0$) or an open part of a product of two plane circles($\lambda_2 \neq 0$).*

Proof. From (2 5) and (2.6) we find

$$x_1^2 + x_2^2 + x_3^2 = \frac{\lambda_2 + 2}{\lambda_2 - \lambda_1} \tag{2-14}$$

$$x_4^2 + x_5^2 = \frac{\lambda_1 + 2}{\lambda_1 - \lambda_2} \tag{2-15}$$

Without loss of generality, we may assume that M^2 is locally given by a set of points

$$(x_1, x_2, f(x_1, x_2), g(x_1, x_2), h(x_1, x_2))$$

or a set of points

$$(x_1, f(x_1, x_4), g(x_1, x_4), x_4, h(x_4)), \tag{2-16}$$

where f, g, h are smooth functions defined on open subsets of E^2 or E^1. In either case we can obtain a local tangent vector field X of M^2 such that

$$DX = \lambda_1 X.$$

For example, if x_1, x_4 are indendent variables, then we have $D\frac{\partial}{\partial x_1} = \lambda_1 \frac{\partial}{\partial x_1}$ from (2.16). Hence we can get a local orthonormal tangent frame e_1, e_2 of M^2 such that

$$De_1 = \lambda_1 e_1. \tag{2-17}$$

Since $\langle De_i, e_j\rangle = -\langle Dx, h(e_i, e_j)\rangle$ and $Dx = h(e_1, e_1) + h(e_2, e_2)$, we get

$$\langle Dx, Dx\rangle = -\langle De_1, e_1\rangle - \langle De_2, e_2\rangle. \tag{2-18}$$

From (2.14) and (2.15) we know that $\langle Dx, Dx\rangle = -\lambda_1\lambda_2 - 2(\lambda_1 + \lambda_2)$. Thus from (2.17) and (2.18) it follows that

$$\langle De_2, e_2\rangle = \lambda_1\lambda_2 + \lambda_1 + 2\lambda_2. \tag{2-19}$$

Let e_3, e_4, e_5 be a local orthonormal normal frame of M^2 such that $e_3 = x$, $e_4 = \frac{1}{\alpha}(Dx + 2x)$, where $\alpha = |Dx + 2x| = \sqrt{-(\lambda_1 + 2)(\lambda_2 + 2)}$. Then the followings hold.

$$De_3 = -2e_3 + \alpha e_4, \tag{2-20}$$

$$De_4 = \alpha e_3 + \beta e_4, \tag{2-21}$$

where $\beta = \lambda_1 + \lambda_2 + 2$. And we have

$$De_2 = \mu e_2 + k e_5, \tag{2-22}$$

$$De_5 = k e_2 + l e_5, \tag{2-23}$$

for some functions μ, k, l. Since tr$D$=$3\lambda_1 + \lambda_2$ and det$D$=$\lambda_1^3\lambda_2^2$, we can see that

$$\mu = \lambda_1\lambda_2 + \lambda_1 + 2\lambda_2, \quad l = -\lambda_1\lambda_2 - \lambda_2, \quad k^2 = \mu l - \lambda_1\lambda_2$$

by (2.17) and (2.20)~(2.23). The coefficients (h_{ij}^r) of the second fundamental form h will be given by

(2-24)

$$[h_{ij}^3] = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad [h_{ij}^4] = \begin{bmatrix} -\frac{\lambda_1+2}{\alpha} & 0 \\ 0 & -\frac{\mu+2}{\alpha} \end{bmatrix}, \quad [h_{ij}^5] = \begin{bmatrix} z & w \\ w & -z \end{bmatrix}.$$

We will show that $k = 0$. Suppose that $k \neq 0$. Differentiating $\langle De_5, e_5 \rangle = l$ in the direction e_1 and using (2.3),(2.24), we have $\langle De_5, -ze_1 - we_2 \rangle = 0$. Using (2.23), from this we can see that $w = 0$. Differentiate (2.17) in the direction e_2. Then by (2.22) and $w = 0$ we find

$$\omega_1^2(e_2)De_2 = \lambda_1 \omega_1^2(e_2)e_2.$$

From this, by (2.22) we get

$$k\omega_1^2(e_2) = 0,\ \omega_1^2(e_2)(\mu - \lambda_1) = 0.$$

This means that $\omega_2^1(e_2) = 0$. Using this and differentiating $\langle De_2, e_2 \rangle = \mu$ in e_2, we have $-zk = 0$. From which we deduce that $z = 0$. Let's differentiate (2.23) in e_1 again. Then we have $k\omega_2^1(e_1) = 0$. And hence we get $\omega_2^1(e_1) = 0$. This imply that the Gaussian curvature $K = \langle h(e_1, e_1), h(e_2, e_2) \rangle - |h(e_1, e_2)|^2$ of M^2 must be zero. So we get $(\lambda_1 + 2)(\lambda_2 - \mu) = 0$. Thus we have $\lambda_1 = -2$ or $\mu = \lambda_2$. If $\lambda_1 = -2$, then (2.15) imply that $x_4 = x_5 = 0$, which yields a contradiction. Thus we must have $\mu = \lambda_2$, which imply that $k = 0$. Thus we may assume that $k = 0$ in (2.22) and (2.23). Therfore $\mu = \lambda_1$ or $\mu = \lambda_2$. If the former holds, then x_4 and x_5 are constants. So M^2 is an open part of 2-sphere. Since M^2 is not minimal in S^4, x_4 or x_5 will be nonzero. This imply that $\lambda_2 = 0$. If $\mu = \lambda_2$, Then M^2 is a product $C_1 \times C_2$, where C_1 and C_2 are curves in S^2 and S^1 respectively. Since $\Delta y = \lambda_1 y$, where y is the position vector of C_1 in S^2, we can see that C_1 is a circle.

3. Proof of theorem

Proof of Theorem. If A is not symmetric, then by Theorem4 in [4] M^2 is contained in 4-dimensional linear subspace of E^5. Hence M^2 is an open part of 2-sphere or an open part of a product of two spheres or a minimal surface of S^3[4]. Now assume that A is symmetric. Then by a suitable coordinate change we may assume that A is diagonal. If $Ax = -2x$ at one point of M^2, then by the constancy of $|H|$ $Ax = -2x$ holds on whole points of M^2 This means that M^2 is a minimal surface of S^4. Thus, if M^2 is not minimal in S^4, then M^2 has no points at which $Ax + 2x$ vanishes. From now on suppose that M^2 is not minimal in S^4. If x, Ax, A^2x are locally linearly dependent on M^2, then by Lemma 1 we can see that M^2 is an open part of a product of two plane circles. Otherwise we may assume that there exist smooth functions α, β on M^2 such that

$$A^2x = \alpha Ax + \beta x. \tag{3-1}$$

Then we have $-2\alpha+\beta = |H|^2$ by (2.1) and $\langle A^2x, x\rangle = |H|^2$. Also since (3.1) imply that $\langle A^2x, AX\rangle = 0$ for any tangent vector X of M^2, we find $\langle A^2x, Ax\rangle = \alpha|H|^2 - 2\beta =$ constant. Thus since $4 - |H|^2 \neq 0$, we know that α and β are constant and from (3.1) we get the following equations:

$$(\lambda_i^2 - \alpha\lambda_i - \beta)x_i = 0, i = 1, \cdots, 5, \tag{3-2}$$

where λ_i are the diagonal entries of A and x_i are the coordinate functions of M^2. Assume that M^2 is locally described as the set of points

$$(x_1, x_2, f(x_1, x_2), g(x_1, x_2), h(x_1, x_2)),$$

where f, g, h are smooth functions defined on an open subset of E^2. Then by (3.2) we can expect the following three cases:

case 1 $\lambda_1 = \lambda_3 \neq \lambda_2 = \lambda_4 \quad and \quad x_5 = 0,$

case 2 $\lambda_1 = \lambda_3 = \lambda_4 \neq \lambda_2 = \lambda_5,$

case 3 $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 \quad and \quad x_5 \quad is\ a\ nonzero\ constant (\lambda_5 = 0).$

If case 1 holds, then we can see that M^2 is an open part of a product of two spheres. If case 2 holds, then M^2 is an open part of 2-sphere or an open part of a product of two plane circles by Lemma 2. And if case 3 holds, then M^2 is a minimal surface of a hypersphere in S^4.

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