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A NOTE ON REAL HYPERSURFACES OF TYPE A IN A COMPLEX SPACE FORM

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1. Introduction

A complex n-dimensional Kähler manifold of constant holomorphic sectional curvature c is called a *complex space form*, which is denoted by $M_n(c)$ A complete and simply connected complex space form consists of a complex projective space $P_n\mathbb{C}$, a complex Euclidean space $\mathbb{C}^n$ or a complex hyperbolic space $H_n\mathbb{C}$, according as $c > 0, c = 0$ or $c < 0$.

In his study of real hypersurfaces of $P_n\mathbb{C}$, Takagi [10] classified all homogeneous real hypersurfaces and Cecil and Ryan [3] showed also that they are realized as the tubes of constant radius over Kähler submanifolds if the structure vector field is principal. And Berndt [2] also classified all homogeneous real hypersurfaces of $H_n\mathbb{C}$ and showed that they are realized as the tubes of constant radius over certain submanifolds. According to Takagi's classification theorem and Berndt's one, the principal curvatures and their multiplicities of homogeneous real hypersurfaces of $M_n(c)$ are given

Now, let M be a real hypersurface of $M_n(c), c \neq 0$. Then M has an almost contact metric structure (ϕ, ξ, η, g) induced from the Kahler metric and the almost complex structure of $M_n(c)$. We denote by A the shape operator in the direction of the unit normal and by ∇ the Riemannain connection on M. Then Okumura [8] and Montiel and Romero [7] proved the following

THEOREM A. *Let M be a real hypersurface of $P_n\mathbb{C}, n \geqq 2$. If it satisfies $A\phi - \phi A = 0$, then M is locally a tube of radius r over one of the following Kähler submanifolds:*

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(A_1) a hyperplane $P_{n-1}\mathbb{C}$, where $0 < r < \pi/2$,
(A_2) a totally geodesic $P_k\mathbb{C}$ $(1 \leqq k \leqq n-2)$, where $0 < r < \pi/2$.

THEOREM B. *Let M be a real hypersurface of $H_n\mathbb{C}, n \geqq 2$. If it satisfies $A\phi - \phi A = 0$, then M is locally one of the following hypersurfaces:*
(A_0) a horosphere in $H_n\mathbb{C}$, i.e., a Montiel tube,
(A_1) a tube of a totally geodesic hyperplane $H_{n-1}\mathbb{C}$,
(A_2) a tube of a totally geodesic $H_k\mathbb{C}$ $(1 \leqq k \leqq n-2)$.

Such real hypersurfaces in Theorems A and B are said to be of *type A*.

Let T_0 be a distribution defined by the subspace $T_0(x) = \{u \in T_xM : g(u, \xi(x)) = 0\}$ of the tangent space T_xM of M at any point x, which is called the *holomorphic distribution*. As an example of non-homogeneous real hypersurfaces in $M_n(c), c \neq 0$, we have ruled real ones. It is also seen in Kimura [5] and Ahn, Lee and Suh [1] that ruled real hypersurfaces are characterized by the holomorphic distribution T_0. On the other hand, for the Hopf fibration $\pi : S^{2n+1}(1) \to P_n\mathbb{C}$, the projection of a hypersurface with parallel second fundamental form in $S^{2n+1}(1)$ becomes a real one in $P_n\mathbb{C}$, which satisfies $\nabla_\xi A = 0$ and $A\xi = 0$. Thus it seems to be interesting the property for $\nabla_\xi A$ restricted to T_0 In his previous paper [9], the first author proved the following theorem.

THEOREM C *Let M be a real hypersurface of $M_n(c), c \neq 0, n \geqq 3$. If it satisfies $\nabla_\xi A(X) = 0$ for any vector field X in T_0 and $g(A\xi, \xi) \neq 0$, then M is of type A.*

In this theorem, the condition $g(A\xi, \xi) \neq 0$ means that the function $g(A\xi, \xi)$ has no zero points.

There is a gap between the conditions $g(A\xi, \xi) \neq 0$ and $A\xi \neq 0$. The purpose of this article is to generalize Theorem C slightly and to take account of the condition $A\xi \neq 0$. We have the following

THEOREM. *Let M be a real hypersurface of $M_n(c), c \neq 0, n \geqq 3$. If it satisfies*

$$\nabla_\xi A(X) = 0, \quad A\xi \neq 0$$

for any vector field X in T_0, then M is of type A.

2. Preliminaries

First of all, we recall fundamental properties of real hypersurfaces of a complex space form. Let M be a real hypersurface of a complex n-dimensional complex space form $M_n(c)$ of constant holomorphic sectional curvature c, and let C be a unit normal vector field on a neighborhood in M. We denote by J the almost complex structure of $M_n(c)$. For a local vector field X on the neighborhood in M, the images of X and C under the linear transformation J can be represented as

$$JX = \phi X + \eta(X)C, \quad JC = -\xi,$$

where ϕ defines a skew-symmetric transformation on the tangent bundle TM of M, while η and ξ denote a 1-form and a vector field on the neighborhood in M, respectively. Then it is seen that $g(\xi, X) = \eta(X)$, where g denotes the Riemannian metric tensor on M induced from the metric tensor on $M_n(c)$. The set of tensors (ϕ, ξ, η, g) is called an *almost contact metric structure* on M. They satisfy the following properties .

$$\phi^2 = -I + \eta \otimes \xi, \quad \phi\xi = 0, \quad \eta(\phi X) = 0, \quad \eta(\xi) = 1$$

for any vector field X, where I denotes the identity transformation Furthermore, the covariant derivatives of the structure tensors are given by

$$\nabla_X \xi = \phi AX, \quad \nabla_X \phi(Y) = \eta(Y)AX - g(AX, Y)\xi \tag{2.1}$$

for any vector fields X and Y on M, where ∇ is the Riemannian connection on M and A denotes the shape operator of M in the direction of C.

Since the ambient space is of constant holomorphic sectional curvature c, the equations of Gauss and Codazzi are respectively obtained :

$$\begin{aligned} R(X,Y)Z = \frac{c}{4}\{&g(Y,Z)X - g(X,Z)Y + g(\phi Y, Z)\phi X \\ &- g(\phi X, Z)\phi Y - 2g(\phi X, Y)\phi Z\} \\ &+ g(AY, Z)AX - g(AX, Z)AY, \end{aligned} \tag{2.2}$$

$$(2.3)\quad \nabla_X A(Y) - \nabla_Y A(X) = \frac{c}{4}\{\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi\},$$

where R denotes the Riemannian curvature tensor of M and $\nabla_X A$ denotes the covariant derivative of the shape operator A with respect to X.

Next, we suppose that the structure vector field ξ is principal with corresponding principal curvature α, namely $A\xi = \alpha\xi$. Then it is seen in [4] and [6] that α is constant on M and it satisfies

$$(2.4)\qquad 2A\phi A = \frac{c}{2}\phi + \alpha(A\phi + \phi A).$$

3. Proof of theorem

Let M be a real hypersurface of $M_n(c), c \neq 0, n \geqq 3$ and assume that

$$(3.1)\qquad \nabla_\xi A(X) = 0$$

for any vector field X in T_0. By the assistance of (2.3), it turns out to be

$$(3.2)\qquad \nabla_Y A(\xi) = -\frac{c}{4}\phi Y$$

for any vector field Y in T_0. Differentiating this equation with respect to X covariantly and taking account of (2.1), we get

$$\nabla_X \nabla_Y A(\xi) + \nabla_{\nabla_X Y} A(\xi) + \nabla_Y A(\phi AX) = \frac{c}{4}\{g(AX, Y)\xi - \phi\nabla_X Y\}$$

for any vector fields X and Y in T_0. Since the component of the vector $\nabla_X Y$ in the direction of ξ is given by $-g(\phi AX, Y)$ by the first equation of (2.1), we have the following orthogonal decomposition

$$\nabla_X Y = (\nabla_X Y)_0 - g(\phi AX, Y)\xi,$$

where $(\nabla_X Y)_0$ denotes the T_0-component of $\nabla_X Y$. By (2.3), (3.1) and the covariant differentiation which is given above and the decomposion, we get

$$\text{(3.3)} \quad \nabla_X \nabla_Y A(\xi) = g(\phi AX, Y)\nabla_\xi A(\xi) - \nabla_Y A(\phi AX) + \frac{c}{4} g(AX, Y)\xi$$

for any vector fields X and Y in T_0. As is well known, the Ricci formula for the shape operator A is given by

$$\nabla_X \nabla_Y A(Z) - \nabla_Y \nabla_X A(Z) = R(X,Y)(AZ) - A(R(X,Y)Z)$$

for any vector fields X, Y and Z. Accordingly, putting $Z = \xi$ and taking X and Y in the distribution T_0, we obtain

$$\text{(3.4)} \quad \begin{aligned} &\nabla_X A(\phi AY) - \nabla_Y A(\phi AX) + g((A\phi + \phi A)X, Y)\nabla_\xi A(\xi) \\ &= \frac{c}{4}\{g(AY,\xi)X - g(AX,\xi)Y + g(A\phi Y,\xi)\phi X \\ &\quad - g(A\phi X,\xi)\phi Y - 2g(\phi X, Y)\phi A\xi\} - g(AY,\xi)A^2 X \\ &\quad + g(AX,\xi)A^2 Y + g(A^2 Y,\xi)AX - g(A^2 X,\xi)AY \end{aligned}$$

by the Gauss quation (2.2) and (3.3).

Now, in order to prove the theorem, we shall suppose that the structure vector field ξ is not principal. Then we can put $A\xi = \alpha\xi + \beta U$, where U is a unit vector field in the holomorphic distribution T_0, and α and β are smooth functions on M. Let M_0 be the non-empty open subset of M consisting of points x at which $\beta(x) \neq 0$. Next, let M_1 be the open subset of M consisting of points x at which $g(A\xi,\xi)(x) \neq 0$. The proof of the theorem is divided two steps. First of all, we asserts

LEMMA 3.1. *The intersection of the subsets M_0 and M_1 is empty.*

Proof. Suppose the intersection $M_0 \cap M_1$ is not empty. Since the subsets M_0 and M_1 are both open and Theorem C is a local property, we can apply Theorem C on $M_0 \cap M_1$ and it has the same property as that of the real hypersurface M of type A, namely, it satisfies that ξ is principal on $M_0 \cap M_1$, which is a contradiction.

It completes the proof.

The second step is to prove that the subset M_0 is empty, namely, the structure vector field ξ is principal.

By Lemma 3.1, the subset M_0 is contained in the complement $M - M_1$ of M_1 in M. So, we see that $\alpha = 0$ and β has no zero points on M_0. Namely, we have $A\xi = \beta U$. Differentiating $A\xi = \beta U$ with respect to X covariantly, we have

$$\nabla_X A(\xi) + A\phi AX = d\beta(X)U + \beta\nabla_X U,$$

and hence it satisfies by (3.2)

$$\beta\nabla_X U = A\phi AX - d\beta(X)U - \frac{c}{4}\phi X \tag{3.5}$$

on M_0.

On the other hand, differentiating $g(A\xi, \xi) = 0$ with respect to X covariantly and taking account of (2 1) and (3 5), we have $g(A\phi U, X) = 0$ for any vector field X in T_0. Since $g(A\phi U, \xi) = 0$, it turns out to be

$$A\phi U = 0 \tag{3.6}$$

on M_0. Differentiating $A\xi = \beta U$ with respect to ξ covariantly, we have by (2.1) and (3.6)

$$\nabla_\xi A(\xi) = d\beta(\xi)U + \beta\nabla_\xi U \tag{3.7}$$

on M_0. Since the vector field $\nabla_\xi U$ is orthogonal to ξ, from (3.7), we see $g(\nabla_\xi A(\xi), \xi) = 0$ on M_0. It implies $\nabla_\xi A(\xi) = 0$ on M_0, from which together with the assumption (3.1), we obtain

$$\nabla_\xi A = 0 \tag{3.8}$$

on M_0. Taking X and Y in T_0, we have

$$g(\nabla_X A(\phi AY), \xi) = -\frac{c}{4}g(AX, Y), \tag{3.9}$$

where we have used (3.2). On the subset M_0, by taking account of (3.8) and (3.9), the inner product of (3.4) with ξ gives us

$$g(X, U)g(Y, AU) - g(Y, U)g(X, AU) = 0,$$

namely, we have

$$g(X,U)AU - g(X,AU)U \equiv 0 \quad (mod.\ \xi) \tag{3.10}$$

for any vector fields X and Y in T_0. Since $g(AU,\xi) = \beta$, we can put $AU = \beta\xi + \gamma U + \delta V$, where V is a unit vector field orthogonal to ξ and U, and γ and δ are smooth functions on M_0. From (3.10), we get

$$\delta\{g(X,U)V - g(X,V)U\} \equiv 0 \quad (mod.\ \xi)$$

for any vector field X in T_0. Thus we have $\delta = 0$, and hence

$$AU = \beta\xi + \gamma U$$

on M_0.

Under the above preperations, we can prove the second step.

LEMMA 3.2. *The structure vector field ξ is principal.*

Proof. By (3.7) and (3.8), we have

$$d\beta(\xi)U + \beta\nabla_\xi U = 0$$

on M_0. The inner product of the above equation with U implies $d\beta(\xi) = 0$. So, we get $\nabla_\xi U = 0$ on M_0 Hence, differentiating $AU = \beta\xi + \gamma U$ with respect to ξ covariantly and taking account of (2.1) and (3.8), we have

$$\beta\phi A\xi + d\gamma(\xi)U = 0,$$

and hence

$$\beta^2\phi U + d\gamma(\xi)U = 0$$

on M_0. This implies that $\beta = 0$ on M_0, which is a contradiction.

It completes the proof.

Finally, from the assumption $A\xi \neq 0$ of the theorem, we have $g(A\xi,\xi) = \alpha \neq 0$ on M_0. Accordingly, we complete the proof of the theorem by Theorem C.

REMARK. The condition that $A\xi$ does not vanish identically on M is a weaker one than the condition that the function $g(A\xi, \xi) = \alpha$ has no zero points. However, there exists an example of non-homogeneous real hypersurface with satisfy the conditions $\nabla_\xi A = 0$ and $A\xi$ vanishes identically. So, the condition that $A\xi \neq 0$ can not be rejected.

As a direct consequence of Theorem, we find the following

COROLLARY 3.1. *Let M be a real hypersurface of $M_n(c), c \neq 0, n \geqq 3$. If the structure vector field ξ is principal and if it satisfies*

$$g(\nabla_\xi A(X), Y) = 0, \quad A\xi \neq 0$$

for any vector fields X and Y in T_0, then M is of type A.

Proof. Since ξ is principal, we see $A\xi = \alpha\xi$ and α is constant. Hence, we have $\nabla_\xi A(\xi) = 0$ because $\nabla_\xi \xi = 0$. From the assumption $g(\nabla_\xi A(X), Y) = 0$ for any vector fields X and Y in T_0, we get $\nabla_\xi A(X) = 0$ for any vector field X in T_0.
This completes the proof.

Now, let $\mathcal{L}_\xi$ be the Lie derivative with respect to ξ. We define the second fundamental form h by $h(X, Y) = g(AX, Y)$ for any vector fields X and Y. Because of $\mathcal{L}_\xi h(X, Y) = g(\nabla_\xi A(X), Y)$ for any vector fields X and Y, we obtain the following corollary by Theorem.

COROLLARY 3.2. *Let M be a real hypersurface of $M_n(c), c \neq 0, n \geqq 3$. If it satisfies*

$$\mathcal{L}_\xi h(X, Y) = 0, \quad A\xi \neq 0$$

for any vector field X in T_0 and any vector field Y, then M is of type A.

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