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PROPERTIES OF PSEUDOCONFORMAL MAPPINGS IN COMPLEX BANACH SPACES

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1. Introduction

T. Higuchi[1] obtained the distribution theorem of holomorphic mappings in several complex variables. P. Liczberski[3] and T. Matsuno[4] investigated the starlikeness of holomorphic mappings in complex vector spaces, separately. And H. J. Kim and K. H. Shon[2] obtained some properties of starlikeness for pseudoconformal mappings in complex Banach spaces. For $(z_1, \cdots, z_n) = z \in \mathbb{C}^n$, define $|z| = \max_{1 \le i \le n} |z_i|$ and let $D_r = \{z \in \mathbb{C}^n : |z| < r\}$ and $D = D_1$ Let $\mathcal{F}$ be the family of $w : D \to \mathbb{C}^n$ which are holomorphic and satisfy $w(0) = 0$, $\mathrm{Re}\left[\dfrac{w_i(z)}{z_i}\right] \ge 0$ when $|z| = |z_i| > 0$, $(1 \le i \le n)$, where $w = (w_1, \cdots, w_n)$.

In this paper, we investigate some properties of starlike mappings with respect to pseudoconformal mappings in complex Banach spaces

2. Preliminaries

Definition 2.1. A holomorphic mapping $f : D \to \mathbb{C}^n$ is starlike if f is univalent, $f(0) = 0$ and $sf(D) \subset f(D)$ for all $s \in I = [0, 1]$.

Definition 2 2 For a system of n holomorphic functions $f_j =$

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$f_j(z)$ $(j = 1, 2, \cdots, n)$, if

$$\det \frac{\partial f}{\partial z} = \begin{vmatrix} \frac{\partial f_1}{\partial z_1} & \cdots & \frac{\partial f_1}{\partial z_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial z_1} & \cdots & \frac{\partial f_n}{\partial z_n} \end{vmatrix} \neq 0$$

then we call f a pseudoconformal mapping.

From Theorems 1 and 2 of T. J Suffridge[5], we have the following theorem.

THEOREM 2.3 *The mapping $f : D \to \mathbb{C}^n$ is starlike if and only if there exists $w \in \mathcal{F}$ such that a pseudoconformal mapping $f = Jw$, where f and w are written as column vectors and $f(0) = 0$.*

DEFINITION 2.4 If $f : D \to \mathbb{C}^n$ is a biholomorphic map of D onto a convex domain, we say that f is convex.

T. J. Suffridge[5] proved that for the pseudoconformal mapping $f : D \to \mathbb{C}^n$ being biholomorphic and $f(0) = 0$, the mapping f is convex if and only if there exists f which is univalent of D onto convex domains such that $f(z) = T(f_1(z_1), f_2(z_2), \cdots, f_n(z_n))$, where T is a nonsingular linear transformation.

3. Starlike mapping in complex Banach spaces

Let X and Y be complex Banach spaces and let $B = \{x \in X : ||x|| < 1\}$. For $0 \neq x \in X$, let $T(x)$ be the collection of all continuous real linear functionals x^* on X satisfying $x^*(x) = x$ and $x^*(y) \leq ||y||$ for all $y \in X$. Let $\mathcal{F}_0(B)$ be the class of mappings $w : B \to X$ which are holomorphic, and satisfy $w(0) = 0$, and $x^*(w(x)) \geq 0$ when $0 \neq x \in B$ and $x^* \in T(x)$. Further let $\mathcal{F}(B)$ be the class of $w \in F_0(B)$ which satisfy $x^*(w(x)) > 0$ when $0 \neq x \in B$ and $x^* \in T(x)$.

We can define a starlike map in the complex Banach spaces like a definition of a starlike map in §2. That is, a holomorphic mapping $f : B \to Y$ is starlike if f is one-to-one , $f(0) = 0$, and $sf(B) \subset f(B)$ for all $s \in I$.

THEOREM 3.1[6]. *Suppose $f : B \to Y$ is starlike and that f^{-1} is holomorphic on an open subset $f(B)$ of Y. There exists $w \in \mathcal{F}(B)$ such that $f(x) = Df(x)w(x)$.*

THEOREM 3.2[6]. *Let $f : B \to Y$ be holomorphic and $f(0) = 0$. Assume $Df(x)$ has a bounded inverse for each $x \in B$ and for some $w \in \mathcal{F}(B)$, $f(x) = Df(x)w(x)$. Then f is starlike.*

EXAMPLE 3.3. Define $f : B \to Y = l^3$ by $f(x) = (ax_1, bx_2, cx_3)$ where a, b, c are arbitrary constants, and $||x||^3 = |x_1|^3 + |x_2|^3 + |x_3|^3$. Then $\dfrac{f(x)}{Df(x)} = w(x)$ where $w(x) = (x_1, x_2, x_3)$ But for $0 \leq t \leq 1$, let $v(x, y, t)$. $B \to B$ be the restriction of the linear map having matrix

$$\begin{pmatrix} 1-t & \sqrt{1-t^2}-1 & \sqrt{1-t^2}-1 \\ \sqrt{1-t^2}-1 & 1-t & \sqrt{1-t^2}-1 \\ \sqrt{1-t^2}-1 & \sqrt{1-t^2}-1 & 1-t \end{pmatrix}.$$

Then f is starlike.

Let $\mathcal{K}_0(B)$ be the class of all functions w $B \times B \times B \to X$ which are holomorphic in each variable and satisfy $w(x, x, x) = 0$ and $x^*(w(x, y, z)) \geq 0$ if $x^* \in T(x)$ and $\max\{||y||, ||z||\} \leq ||x||$. Let $\mathcal{K}(B)$ be the colletion of all $w \in \mathcal{K}_0(B)$ which satisfy $x^*(w(x, y, z)) > 0$ when $x^* \in T(x)$ and $\max\{||y||, ||z||\} < ||x||$. The technique of the following theorem is based on the method in T J Suffridge[6].

THEOREM 3.4 *If $w \in \mathcal{K}_0(B)$ and $|\alpha| < 1$ then $\dfrac{1}{\alpha}w(\alpha x, \alpha y, \alpha z) \in \mathcal{K}_0(B)$ (the limit value at $\alpha = 0$ is $Dw(0, 0, 0)(x, y, z)$). Furthermore if $x^* \in T(x)$, $0 \neq x \in B$ and $\max\{||y||, ||z||\} \leq ||x||$, then $x^*(w(x, y, z)) = 0$ if and only if $x^*(Dw(0, 0, 0)) = 0$.*

Proof. For $0 < |\alpha| < 1$, $x^* \in T(x)$, define x^*_α by

$$x^*_\alpha((x, y, z)) = x^* \left(|\alpha| \frac{(x, y, z)}{\alpha} \right)$$

for all $(x, y, z) \in X \times X \times X$. Then $x^*_\alpha \in T(\alpha x)$. Thus,

$$0 \leq \frac{1}{|\alpha|} x^*_\alpha(w(\alpha x, \alpha y, \alpha z)) = \frac{1}{|\alpha|} x^* \left(|\alpha| \frac{w(\alpha x, \alpha y, \alpha z)}{\alpha} \right)$$

$$= x^* \left(\frac{w(\alpha x, \alpha y, \alpha z)}{\alpha} \right).$$

Since x^* is continuous, we have

$$\frac{1}{\alpha} w(\alpha x, \alpha y, \alpha z) \in K_0(B)$$

for $|\alpha| < 1$. Since $x^*((x, y, z)) = \mathrm{Re}[x^*((x, y, z) - ix^*(i(x, y, z))]$ is the real part of a continuous complex linear functional

$$x^* \left(\frac{w(\alpha x, \alpha y, \alpha z)}{\alpha} \right)$$

is nonnegative harmonic of α for fixed (x, y, z) and $|\alpha| < \frac{1}{||(x, y, z)||}$. Since

$$\frac{1}{\alpha} w(\alpha x, \alpha y, \alpha z) \in K_0(B),$$

we have

$$x^* \left(\frac{(\alpha x, \alpha y, \alpha z)}{\alpha} \right) \geq 0$$

if $x^* \in T(x)$. Hence w is holomorphic and so

$$x^* \left(\frac{w(\alpha x, \alpha y, \alpha z)}{\alpha} \right)$$

$$= \mathrm{Re} \left[x^* \left(\frac{w(\alpha x, \alpha y, \alpha z)}{\alpha} \right) - ix^* \left(i \frac{w(\alpha x, \alpha y, \alpha z)}{\alpha} \right) \right]$$

is harmonic Therefore

$$x^* \left(\frac{w(\alpha x, \alpha y, \alpha z)}{\alpha} \right) > 0$$

or

$$x^* \left(\frac{w(\alpha x, \alpha y, \alpha z)}{\alpha} \right) \equiv 0$$

for fixed (x, y, z). Hence we have $x^*(Dw(0, 0, 0)(x, y, z)) \equiv 0$.

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