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FIXED POINTS OF ASYMPTOTICALLY REGULAR MAPPINGS

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ABSTRACT In this paper, we prove some common fixed point theorems for compatible mappings by using asymptotically regular mappings under the contractive type of G. E Hardy and T D Rogers, and also give some examples to illustrate our main theorems Our results extend the results of M D. Guay and K L Singh and others

1. Introduction

The most well-known fixed point theorem is so called the *Banach's fixed point theorem* which asserts that if a contractive mapping from a complete metric space into itself exists, then the mapping has a unique fixed point in a complete metric space. A more generalized contractive condition was introduced by G. E Hardy and T. D. Rogers [4].

In 1976, G Jungck [5] initially proved a common fixed point theorem for commuting mappings which generalizes the well-known Banach's fixed point theorem. This result has been generalized, extended and improved in various ways by many authors ([2], [6]-[12])

On the other hand, S. Sessa [11] introduced a generalization of commutativity, which is called weak commutativity, and proved some common fixed point theorems for weakly commuting mappings which generalize the results of K M. Das and K. V. Naik [2]. Recently, G Jungck [6] introduced the concept of more generalized commutativity, so called compatibility, which is more general than that of weak commutativity The utility of compatibility in the context of fixed point theory was initially demonstrated in extending a theorem of S Park and J S. Bae

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[10]. By employing compatible mappings in stead of commuting mappings and using four mappings in stead of three mappings, G. Jungck [7] extended the results of M. S. Khan and M. Imdad [9], S. L. Singh and S. P. Singh [12] and, recently, also obtained an interesting result concerning with his concept in his consecutive paper [8].

In this paper, we prove some common fixed point theorems for compatible mappings by using asymptotically regular mappings under the contractive type of G. E. Hardy and T. D. Rogers [4], and also give some examples to illustrate our main theorems. Our results extend the results of M. D. Guay and K. L. Singh [3] and others.

2. Preliminaries

For some definitions, terminologies and notations in this paper, we refer to [6], [7], and [11].

DEFINITION 2.1. Let A and B be mappings from a metric space (X, d) into itself. Then A and B are said to be *weakly commuting mappings* on X if $d(ABx, BAx) \leq d(Ax, Bx)$ for all x in X

Clearly, commuting mappings are weakly commuting, but the converse is not necessarily true as in the following example:

EXAMPLE 2.1. Let $X = [0, 1]$ with the Euclidean metric d. Define A and $B : X \to X$ by

$$Ax = \frac{1}{2}x \quad \text{and} \quad Bx = \frac{x}{2+x}$$

for all x in X, respectively. Then A and B are weakly commuting mappings on X, but they are not commuting at x $(\neq 0)$ in X.

DEFINITION 2.2. Let A and B be mappings from a metric space (X, d) into itself. Then A and B are said to be *compatible mappings* on X if $\lim_{n\to\infty} d(ABx_n, BAx_n) = 0$ when $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = t$ for some t in X.

Obviously, weakly commuting mappings are compatible, but the converse is not necessarily true as in the following example:

EXAMPLE 2.2. Let $X = (-\infty, \infty)$ with the Euclidean metric d. Define A and $B : X \to X$ by

$$Ax = x^3 \quad \text{and} \quad Bx = 2 - x$$

for all x in X, respectively. Then, since $d(Ax_n, Bx_n) = |x_n - 1| \, |x_n^2 + x_n + 2| \to 0$ if and only if $x_n \to 1$,

$$\lim_{n\to\infty} d(BAx_n, ABx_n) = \lim_{n\to\infty} 6\,|x_n - 1|^2 = 0 \quad \text{as } x_n \to 1.$$

Thus, A and B are compatible on X, but they are not weakly commuting mappings at x $(= 0)$ in X. Thus, commuting mappings are also compatible, but the converse is not necessarily true.

We need the following lemma for our main theorems:

LEMMA 2 1. *Let A and B be compatible mappings from a metric space (X, d) into itself. Suppose that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = t$ for some t in X. Then $\lim_{n\to\infty} BAx_n = At$ if A is continuous.*

3. Fixed point theorems

In this section, we shall prove some fixed point theorems for asymptotically regular mappings.

DEFINITION 3.1. Let A and S be mappings from a metric space (X, d) into itself Then A is said to be *asymptotically regular* at x_0 in X with respect to S if $\lim_{n\to\infty} d(SA^n x_0, A^{n+1}x_0) = 0$

If S is the identity mapping on X, then the above Definition 3 1 coincides with that of F. E. Browder and W. V. Petryshyn [1].

Drawing inspiration from the contractive condition of G E. Hardy and T. D. Rogers [4], we present the following theorems.

THEOREM 3 1. *Let A, S and T be mappings from a complete metric space (X, d) into itself satisfying the following condition:*

$$\begin{aligned} d(Ax, Ay) \le\ & a_1\, d(Ax, Sx) + a_2\, d(Ax, Tx) + a_3\, d(Ay, Sy) \\ & + a_4\, d(Ay, Ty) + a_5\, d(Ax, Sy) + a_6\, d(Ax, Ty) \\ & + a_7\, d(Ay, Sx) + a_8\, d(Ay, Tx) + a_9\, d(Sx, Ty) \\ & + a_{10}\, d(Sy, Tx) \end{aligned} \tag{3.1}$$

for all x, y in X, where a_i, $i = 1, 2, \cdots, 10$, are non-negative real numbers with

$$\begin{aligned}(3.2) \qquad &\max\{(a_2 + a_3 + a_6 + a_7 + a_9 + a_{10}), (a_3 + a_4 + a_7 + a_8), \\ &\qquad (a_5 + a_6 + a_7 + a_8 + a_9 + a_{10})\} < 1.\end{aligned}$$

Suppose that

(3.3) S and T are continuous,

(3.4) *the pairs A, S and A, T are compatible,*

(3.5) *A is asymptotically regular at some point x_0 in X with respect to S and T.*

Then A, S and T have a unique common fixed point in X.

Proof. Let A be asymptotically regular at x_0 in X with respect to S and T. Consider the sequence $\{A^n x_0\}$. Then by (3.1), we have for any positive integers n and m,

$$\begin{aligned} d(A^n x_0, A^m x_0) \leq\ & a_1\, d(A^n x_0, SA^{n-1}x_0) + a_2\, d(A^n x_0, TA^{n-1}x_0) \\ & + a_3\, d(A^m x_0, SA^{m-1}x_0) + a_4\, d(A^m x_0, TA^{m-1}x_0) \\ & + a_5\, [\, d(A^n x_0, A^m x_0) + d(A^m x_0, SA^{m-1}x_0)\,] \\ & + a_6\, [\, d(A^n x_0, A^m x_0) + d(A^m x_0, TA^{m-1}x_0)\,] \\ & + a_7\, [\, d(A^m x_0, A^n x_0) + d(A^n x_0, SA^{n-1}x_0)\,] \\ & + a_8\, [\, d(A^m x_0, A^n x_0) + d(A^n x_0, TA^{n-1}x_0)\,] \\ & + a_9\, [\, d(SA^{n-1}x_0, A^n x_0) + d(A^n x_0, A^m x_0) \\ & \qquad + d(A^m x_0, TA^{m-1}x_0)\,] \\ & + a_{10}\, [\, d(SA^{m-1}x_0, A^m x_0) + d(A^m x_0, A^n x_0) \\ & \qquad + d(A^n x_0, TA^{n-1}x_0)\,], \end{aligned}$$

where $a_i \geq 0$, $i = 1, 2, \cdots, 10$. Therefore, we obtain

$$\begin{aligned} (1 - a_5 - a_6 - a_7 - a_8 - a_9 - a_{10})\, & d(A^n x_0, A^m x_0) \\ \leq\ & (a_1 + a_7 + a_9)\, d(A^n x_0, SA^{n-1}x_0) \\ & + (a_2 + a_8 + a_{10})\, d(A^n x_0, TA^{n-1}x_0) \\ & + (a_3 + a_5 + a_{10})\, d(A^m x_0, SA^{m-1}x_0) \\ & + (a_4 + a_6 + a_9)\, d(A^m x_0, TA^{m-1}x_0). \end{aligned}$$

From (3.2) and (3.5), by taking the limit as $m,\ n \to \infty$, $\{A^n x_0\}$ is a Cauchy sequence in X and hence it converges to some point z in X.

$$d(SA^{n-1}x_0, z) \leq d(SA^{n-1}x_0, A^n x_0) + d(A^n x_0, z) \ \to \ 0$$

as $n \to \infty$, so $SA^{n-1}x_0 \to z$. Similarly, we have $TA^{n-1}x_0 \to z$ as $n \to \infty$. By (3 3), we obtain

$$\begin{aligned} &SA^n x_0, \quad S^2A^{n-1}x_0 \ \text{ and } \ STA^{n-1}x_0 \ \to \ Sz, \\ &TA^n x_0, \quad T^2A^{n-1}x_0 \ \text{ and } \ TSA^{n-1}x_0 \ \to \ Tz. \end{aligned}$$

as $n \to \infty$. By (3.4), it follows from Lemma 2.1 that

$$ASA^{n-1}x_0 \ \to \ Sz \quad \text{and} \quad ATA^{n-1}x_0 \ \to \ Tz$$

as $n \to \infty$. From (3.1) with $a_i \geq 0$, $i = 1, 2, \cdots, 10$, we have

$$\begin{aligned} &d(ASA^{n-1}x_0, ATA^{n-1}x_0) \\ &\leq a_1\, d(ASA^{n-1}x_0, S^2A^{n-1}x_0) + a_2\, d(ASA^{n-1}x_0, TSA^{n-1}x_0) \\ &\quad + a_3\, d(ATA^{n-1}x_0, STA^{n-1}x_0) + a_4\, d(ATA^{n-1}x_0, T^2A^{n-1}x_0) \\ &\quad + a_5\, d(ASA^{n-1}x_0, STA^{n-1}x_0) + a_6\, d(ASA^{n-1}x_0, T^2A^{n-1}x_0) \\ &\quad + a_7\, d(ATA^{n-1}x_0, S^2A^{n-1}x_0) + a_8\, d(ATA^{n-1}x_0, TSA^{n-1}x_0) \\ &\quad + a_9\, d(S^2A^{n-1}x_0, T^2A^{n-1}x_0) + a_{10}\, d(STA^{n-1}x_0, TSA^{n-1}x_0) \\ &\leq a_1\, d(ASA^{n-1}x_0, S^2A^{n-1}x_0) \\ &\quad + (a_2 + a_3 + a_6 + a_7 + a_9 + a_{10}) \max\{d(ASA^{n-1}x_0, TSA^{n-1}x_0), \\ &\qquad d(ATA^{n-1}x_0, STA^{n-1}x_0), d(ASA^{n-1}x_0, T^2A^{n-1}x_0), \\ &\qquad d(ATA^{n-1}x_0, S^2A^{n-1}x_0), d(S^2A^{n-1}x_0, T^2A^{n-1}x_0), \\ &\qquad d(STA^{n-1}x_0, TSA^{n-1}x_0)\} + a_4\, d(ATA^{n-1}x_0, T^2A^{n-1}x_0) \\ &\quad + a_5\, d(ASA^{n-1}x_0, STA^{n-1}x_0) + a_8\, d(ATA^{n-1}x_0, TSA^{n-1}x_0). \end{aligned}$$

By taking $n \to \infty$, we have

$$d(Sz, Tz) \leq (a_2 + a_3 + a_6 + a_7 + a_9 + a_{10})\, d(Sz, Tz)$$

which, by (3.2), implies that $Sz = Tz$. Again, from (3.1) with $a_i \geq 0$, $i = 1, 2, \cdots, 10$, we obtain

$$\begin{aligned}
&d(ATA^{n-1}x_0, Az) \\
&\leq a_1\, d(ATA^{n-1}x_0, STA^{n-1}x_0) + a_2\, d(ATA^{n-1}x_0, T^2A^{n-1}x_0) \\
&\quad + a_3\, [\, d(Az, Tz) + d(Tz, Sz)\,] + a_4\, d(Az, Tz) \\
&\quad + a_5\, d(ATA^{n-1}x_0, Sz) + a_6\, d(ATA^{n-1}x_0, Tz) \\
&\quad + a_7\, d(Az, STA^{n-1}x_0) + a_8\, d(Az, T^2A^{n-1}x_0) \\
&\quad + a_9\, d(STA^{n-1}x_0, Tz) + a_{10}\, d(Sz, T^2A^{n-1}x_0) \\
&\leq a_1\, d(ATA^{n-1}x_0, STA^{n-1}x_0) + a_2\, d(ATA^{n-1}x_0, T^2A^{n-1}x_0) \\
&\quad + (a_3 + a_4 + a_7 + a_8) \max\{d(Az, Tz), d(Az, STA^{n-1}x_0), \\
&\qquad d(Az, T^2A^{n-1}x_0)\} + a_5\, d(ATA^{n-1}x_0, Tz) \\
&\quad + a_6\, d(ATA^{n-1}x_0, Sz) + a_9\, d(STA^{n-1}x_0, Sz) \\
&\quad + a_{10}\, d(Tz, T^2A^{n-1}x_0).
\end{aligned}$$

By taking $n \to \infty$, we have

$$d(Tz, Az) \leq (a_3 + a_4 + a_7 + a_8)\, d(Tz, Az)$$

which implies that $Tz = Az$ From (3.1) with $a_i \geq 0$, $i = 1, 2, \cdots, 10$, we obtain

$$\begin{aligned}
&d(A^n x_0, Az) \\
&\leq a_1\, d(A^n x_0, SA^{n-1}x_0) + a_2\, d(A^n x_0, TA^{n-1}x_0) \\
&\quad + a_3\, d(Az, Sz) + a_4\, d(Az, Tz) + a_5\, d(A^n x_0, Sz) \\
&\quad + a_6\, d(A^n x_0, Tz) + a_7\, d(Az, SA^{n-1}x_0) \\
&\quad + a_8\, d(Az, TA^{n-1}x_0) + a_9\, d(SA^{n-1}x_0, Tz) \\
&\quad + a_{10}\, d(Sz, TA^{n-1}x_0) \\
&\leq a_1\, d(A^n x_0, SA^{n-1}x_0) + a_2\, d(A^n x_0, TA^{n-1}x_0) \\
&\quad + (a_5 + a_6 + a_7 + a_8 + a_9 + a_{10}) \max\{d(Az, SA^{n-1}x_0), \\
&\qquad d(Az, TA^{n-1}x_0), d(A^n x_0, Az)\}.
\end{aligned}$$

Thus, by taking $n \to \infty$, we have

$$d(z, Az) \leq (a_5 + a_6 + a_7 + a_8 + a_9 + a_{10})\, d(z, Az)$$

which implies $z = Az$ and hence z is a common fixed point of A, S and T.

In order to prove the uniqueness of z, let w be another common fixed points of A, S and T. Then by (3.1), we obtain

$$\begin{aligned} d(z,w) &= d(Az, Aw) \\ &\leq a_1\, d(Az, Sz) + a_2\, d(Az, Tz) + a_3\, d(Aw, Sw) \\ &\quad + a_4\, d(Aw, Tw) + a_5\, d(Az, Sw) + a_6\, d(Az, Tw) \\ &\quad + a_7\, d(Aw, Sz) + a_8\, d(Aw, Tz) + a_9\, d(Sz, Tw) \\ &\quad + a_{10}\, d(Sw, Tz) \\ &= (a_5 + a_6 + a_7 + a_8 + a_9 + a_{10})\, d(z, w), \end{aligned}$$

where $a_i \geq 0$, $i = 1, 2, \cdots, 10$. Hence $z = w$ by (3.2). This completes the proof.

In the following example, we show that the continuity of both mappings is essential.

EXAMPLE 3.1. Let $X = [0, 1]$ with the Euclidean metric d. Define A, S and $T : X \to X$ by

$$Ax = \frac{1}{8}x, \quad Sx = \frac{1}{2}x \quad \text{and} \quad Tx = \begin{cases} \dfrac{1}{2} & \text{if } x = 0, \\ \dfrac{1}{2}x & \text{if } x \neq 0 \end{cases}$$

for all x in X. Then we have

$$d(AT0, TA0) = \frac{1}{2} - \frac{1}{16} = \frac{7}{16} < \frac{1}{2} = d(A0, T0)$$

and so each of the pairs A, S and A, T is compatible. Furthermore, we have

$$d(Ax,Ay)=\begin{cases} 0\le \frac{1}{4}\,d(Sy,Tx) & \text{if } x=y=0,\\ \frac{1}{8}\,x=\frac{1}{4}\,d(Sy,Tx) & \text{if } x>y=0,\\ \frac{1}{8}\,y=\frac{1}{4}\,d(Sx,Ty) & \text{if } y>x=0,\\ \frac{1}{8}\,|x-y|=\frac{1}{4}\,d(Sx,Ty) & \text{if } x,y\neq 0.\end{cases}$$

Thus, the conditions (3.1) and (3.2) are satisfied with $a_i = 0$, $i = 1,2,\cdots,8$, and $a_9 = a_{10} = \frac{1}{4}$. We have

$$\lim_{n\to\infty} d(A^{n+1}x_0, SA^nx_0) = 0, \qquad \lim_{n\to\infty} d(A^{n+1}x_0, TA^nx_0) = 0$$

at some point x_0 in X. Thus, all the hypotheses of Theorem 3.1 are satisfied except the continuity of both S and T. Here, A, S and T do not have a common fixed point in X.

If we put $S = T$ in Theorem 3.1, then we have the following:

COROLLARY 3.2. *Let A and S be mappings from a complete metric space (X,d) into itself satisfying the following condition:*

$$\begin{aligned} d(Ax,Ay) \le b_1\,d(Ax,Sx) + b_2\,d(Ay,Sy)\\ + b_3\,d(Ax,Sy) + b_4\,d(Ay,Sx) + b_5\,d(Sx,Sy)\end{aligned}$$

for all x,y in X, where b_i, $i = 1,2,\cdots,5$, are non-negative real numbers with

$$b_1 + b_2 + b_3 + b_4 + b_5 < 1. \tag{3.6}$$

Suppose that
(3.7) *S is continuous,*
(3.8) *the pair A,S is compatible,*
(3.9) *A is asymptotically regular at some point x_0 in X with respect to S.*
Then A and S have a unique fixed point in X.

Now, we give some theorems by using four mappings in stead of three mappings:

THEOREM 3.3. *Let A, B, S and T be mappings from a complete metric space (X,d) into itself satisfying the following conditions (3.7), (3.8) and (3.10):*

$$
\begin{aligned}
d(Ax,By) \le\ & b_1\, d(Ax,Sx) + b_2\, d(By,Ty) \\
& + b_3\, d(Ax,Ty) + b_4\, d(By,Sx) + b_5\, d(Sx,Ty)
\end{aligned}
\tag{3.10}
$$

for all x, y in X, where b_i, $i = 1, 2, \cdots, 5$, are non-negative real numbers with

$$
\max\{(b_1+b_3),\ (b_2+b_4),\ (b_3+b_4+b_5)\} < 1 \tag{3.11}
$$

Suppose that

(3.12) *$d(x,Tx) \le d(x,Sx)$ for all x in X,*

(3.13) *A and B are asymptotically regular at some points x_0 and y_0 in X with respect to S and T, respectively.*

Then A, B, S and T have a unique common fixed point in X.

Proof. Let A and B be asymptotically regular at x_0 and y_0 in X with respect to S and T, respectively. Consider the sequences $\{A^n x_0\}$ and $\{B^n y_0\}$. From (3.10), we have, for any positive integers m and n,

$$
\begin{aligned}
d(A^m x_0, B^n y_0) \le\ & b_1\, d(A^m x_0, SA^{m-1}x_0) + b_2\, d(B^n y_0, TB^{n-1}y_0) \\
& + b_3 \,[\, d(A^m x_0, B^n y_0) + d(B^n y_0, TB^{n-1}y_0) \,] \\
& + b_4 \,[\, d(B^n y_0, A^m x_0) + d(A^m x_0, SA^{m-1}x_0) \,] \\
& + b_5 \,[\, d(SA^{m-1}x_0, A^m x_0) + d(A^m x_0, B^n y_0) \\
& \qquad + d(B^n y_0, TB^{n-1}y_0) \,],
\end{aligned}
$$

where $b_i \ge 0$, $i = 1, 2, \cdots, 5$, and so

$$
\begin{aligned}
&(1-b_3-b_4-b_5)\, d(A^m x_0, B^n y_0) \\
&\qquad \le (b_1+b_4+b_5)\, d(A^m x_0, SA^{m-1}x_0) \\
&\qquad\quad + (b_2+b_3+b_5)\, d(B^n y_0, TB^{n-1}y_0).
\end{aligned}
$$

Since $d(A^m x_0, A^n x_0) \le d(A^m x_0, B^n y_0) + d(A^n x_0, B^n y_0)$, we obtain

$$
\begin{aligned}
&d(A^m x_0, A^n x_0)\\
&\le \frac{b_1 + b_4 + b_5}{1-(b_3+b_4+b_5)}\, d(A^m x_0, SA^{m-1}x_0)\\
&\quad + \frac{b_1' + b_4' + b_5'}{1-(b_3'+b_4'+b_5')}\, d(A^n x_0, SA^{n-1}x_0)\\
&\quad + \left(\frac{b_2 + b_3 + b_5}{1-(b_3+b_4+b_5)} + \frac{b_2' + b_3' + b_5'}{1-(b_3'+b_4'+b_5')}\right) d(B^n y_0, TB^{n-1}y_0).
\end{aligned}
$$

where b_i, $b_i' \ge 0$, $i = 1, 2, \cdots, 5$. From (3.11) and (3.13), by taking the limit as $m, n \to \infty$, we deduce that $\{A^n x_0\}$ is a Cauchy sequence in X and hence it converges to a point z in X. Thus,

$$d(SA^{n-1}x_0, z) \le d(A^n x_0, SA^{n-1}x_0) + d(A^n x_0, z) \;\to\; 0$$

as $n \to \infty$ and so $SA^{n-1}x_0 \to z$. Similarly, it can be proved that $B^n y_0$ and $TB^{n-1}y_0 \to z$ as $n \to \infty$. By (3.7), we have

$$SA^n x_0, \quad S^2A^{n-1}x_0 \;\to\; Sz \quad \text{as } n \to \infty.$$

Since A and S are compatible, Lemma 2.1 implies

$$ASA^{n-1}x_0 \;\to\; Sz \quad \text{as } n \to \infty.$$

Again, from (3.10) with $b_i \ge 0$, $i = 1, 2, \cdots, 5$, we obtain

$$
\begin{aligned}
&d(ASA^{n-1}x_0, B^n y_0)\\
&\le (b_1 + b_2)\max\{d(ASA^{n-1}x_0, S^2A^{n-1}x_0), d(B^n y_0, TB^{n-1}y_0)\}\\
&\quad + (b_3 + b_4 + b_5)\max\{d(ASA^{n-1}x_0, TB^{n-1}y_0),\\
&\qquad d(B^n y_0, S^2A^{n-1}x_0), d(S^2A^{n-1}x_0, TB^{n-1}y_0)\}
\end{aligned}
$$

By taking the limit as $n \to \infty$, we have

$$d(Sz, z) \le (b_3 + b_4 + b_5)\, d(Sz, z)$$

and so $Sz = z$ by (3.11). From (3.10) with $b_i \geq 0$, $i = 1, 2, \cdots, 5$, we obtain

$$\begin{aligned} d(Az, B^n y_0) \leq (b_1 + b_3) \max\{d(Az, Sz), d(Az, TB^{n-1} y_0)\} \\ + (b_2 + b_4 + b_5) \max\{d(B^n y_0, TB^{n-1} y_0), \\ d(B^n y_0, Sz), d(Sz, TB^{n-1} y_0)\} \end{aligned}$$

By taking $n \to \infty$, we have

$$\begin{aligned} d(Az, z) &\leq (b_1 + b_3) \max\{d(Az, Sz), d(Az, z)\} \\ &\quad + (b_2 + b_4 + b_5)\, d(z, Sz) \\ &= (b_1 + b_3)\, d(Az, z), \end{aligned}$$

which implies $Az = z$ by (3.11). From (3.12), we deduce that

$$d(z, Tz) \leq d(z, Sz) = 0.$$

Thus, $z = Tz$. Furthermore, (3.10) implies that

$$\begin{aligned} d(z, Bz) &= d(Az, Bz) \\ &\leq b_1\, d(Az, Sz) + b_2\, d(Bz, Tz) \\ &\quad + b_3\, d(Az, Tz) + b_4\, d(Bz, Sz) + b_5\, d(Sz, Tz) \\ &= (b_2 + b_4)\, d(z, Bz), \end{aligned}$$

where $b_i \geq 0$, $i = 1, 2, \cdots, 5$. By (3.11), we have $Bz = z$. Therefore, z is a common fixed point of A, B, S and T.

In order to prove the uniqueness of z, let w be another common fixed point of A and S. Then by (3.10), we obtain

$$\begin{aligned} d(w, z) &= d(Aw, Bz) \\ &\leq b_1\, d(Aw, Sw) + b_2\, d(Bz, Tz) \\ &\quad + b_3\, d(Aw, Tz) + b_4\, d(Bz, Sw) + b_5\, d(Sw, Tz) \\ &= (b_3 + b_4 + b_5)\, d(z, w), \end{aligned}$$

where $b_i \geq 0$, $i = 1, 2, \cdots, 5$. Hence $z = w$. Similarly, we prove that z is a unique common fixed point of B and T. This completes the proof.

REMARK 3.1. Note that, if we do not assume the condition (3.12), i.e., "$d(x,Tx) \leq d(x,Sx)$ for all x in X", in Theorem 3.3, then this theorem need no longer be true. In the following example, we can construct mappings A, B, S and T, where S and T do not satisfy the condition mentioned above, such that A and S have only one common fixed point in X which is not the fixed point of B or T.

EXAMPLE 3.2. Let $X = [0,1]$ with the Euclidean metric d. Define A, B, S and $T : X \to X$ by

$$Ax = 0, \;\; Bx = \begin{cases} \dfrac{1}{4} & \text{if } x = 0, \\ \dfrac{1}{4} & \text{if } x \neq 0, \end{cases} \quad Sx = x \;\text{ and }\; Tx = \begin{cases} 1 & \text{if } x = 0, \\ x & \text{if } x \neq 0 \end{cases}$$

for all x in X, respectively. It is easily seen that A commutes with S and S is continuous. Since $d(0,T0) = 1 > 0 = d(0,S0)$, the condition (3.12) is not satisfied at the point zero. Furthermore, we have

$$d(Ax,By) = \begin{cases} \dfrac{1}{4} = \dfrac{1}{3}\, d(By,Ty) & \text{if } y = 0, \\ \dfrac{1}{4}\, y = \dfrac{1}{3}\, d(By,Ty) & \text{if } y \neq 0 \end{cases}$$

for all x, y in X. For some x_0 and y_0 in X,

$$\lim_{n\to\infty} d(A^{n+1}x_0, SA^n x_0) = 0, \quad \lim_{n\to\infty} d(B^{n+1}y_0, TB^n y_0) = 0.$$

Thus, all the hypotheses of Theorem 3.3 are satisfied with $b_i = 0$, $i = 1,3,4,5$, and $b_2 = \frac{1}{3}$ except the condition (3.12). Here, the point 0 is the common fixed point of A and S but the point 0 is not a fixed point of B and T.

REMARK 3.2. A result which is analogous to Theorem 3.3 can be reformulated by supposing the continuity of T, "$d(x,Sx) \leq d(x,Tx)$ for all x in X" and requiring the compatiblity of the pair B, T.

The following Corollary 3.4 follows easily from Theorem 3.3 and Remark 3.2:

COROLLARY 3.4. *Let A, B, S and T be mappings from a complete metric space (X, d) into itself satisfying the conditions* (3.3), (3.13) *and* (3 10), *where b_i, $i = 1, 2, \cdots, 5$ are non-negative real numbers with* (3.11). *Suppose that*
(3.13) *the pairs A, S and B, T are compatible.*
Then A, B, S and T have a unique common fixed point in X.

We show that the condition of asymptotic regularity is also necessary in Theorem 3.1 and Corollary 3.4:

EXAMPLE 3 3. Let $X = [1, \infty)$ with the Euclidean metric d. Define A $(= B)$ and $S = T : X \to X$ by

$$Ax = 2x \quad \text{and} \quad Sx = 4x$$

for all x in X, respectively Then the conditions (3.1) and (3.2) are satisfied with $a_i = 0$, $i = 1, 2, \cdots, 8$, and $a_9 = a_{10} = h$, where $\frac{1}{4} \leq h < \frac{1}{2}$. Furhermore, the conditions (3.10) and (3 11) are satisfied with $b_i = 0$, $i = 1, 2, 3, 4$ and $b_5 = k$, where $\frac{1}{2} \leq k < 1$ The other hypotheses of Theorem 3.1 and Corollary 3.4 are satisfied except the condition (3.5) and (3.13), respectively. Indeed, for some x_0 in X,

$$\lim_{n \to \infty} d(A^{n+1}x_0, SA^n x_0) = 0 \quad \text{iff} \quad x_0 = 0$$

but the point 0 does not belong to X. Here, none of mappings has a fixed point in X.

REMARK 3 3. If we replace the condition (3.11) by the condition (3.6), then the conclusion of Corollary 3.4 is still true.

REMARK 3.4 If we put $A = B$ and $S = T$ in Corollary 3.4, by Remark 3.3, Corollary 3.4 induces to Corollary 3.2.

REMARK 3 5. If we put $S = I_X$ (: the identity mapping on X) in Corollary 3 4 with $b_1 = b_2$ and $b_3 = b_4$, then we obtain the results of M. D Guay and K. L. Singh [3] which generalize the results of several authors.

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