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A PROOF OF THE LEGENDRE DUPLICATION FORMULA FOR THE GAMMA FUNCTION

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ABSTRACT There have been various proofs of the Legendre duplication formula for the Gamma function Another proof of the formula is given here and a brief history of the Gamma function is also provided

1. Introduction

The birth of the Gamma function was seen in two letters from Leonhard Euler (1707-1783) to Christian Goldbach (1690-1764), just as the simple desire to extend factorials to values between the integers The first letter dated October 13, 1729 dealt with the interpolation problem, while the second dated January 8, 1730 dealt with integration and tied the two together. Euler gave us the well-known Gamma function

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1}\, dt, \qquad \mathrm{Re}(z) > 0, \tag{1.1}$$

where the notation Γ is due, in fact, to Adrin Marie Legendre (1752-1833). Euler considered z as the positive real numbers in (1.1) and the move to the complex plane was initiated by Carl Friedrich Gauss (1777-1855). Legendre calls the integral (1.1) the second Eulerian integral. The first Eulerian integral is currently known as the Beta function and is now conventionally written

$$B(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1}\, dt, \tag{1.2}$$

where $\mathrm{Re}(p) > 0$ and $\mathrm{Re}(q) > 0$. The Gamma function satisfies the relationships

$$\begin{aligned} &\Gamma(z+1) = z\Gamma(z),\ \Gamma(n+1) = n!\ (n = 0, 1, 2, \ldots), \text{ and} \\ &\Gamma(1/2) = \sqrt{\pi} \end{aligned} \tag{1.3}$$

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There is the well-known relationship between Euler's two types of integrals (see Choi and Nam [2]):

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}, \qquad \text{and so} \quad B(p,q) = B(q,p). \tag{1.4}$$

The Legendre duplication formula for the Gamma function is given as follows:

$$\sqrt{\pi}\Gamma(2z) = 2^{2z-1}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right) \tag{1.5}$$

which was discovered by Legendre and extended by Gauss (see Magnus et al. [3, p. 3]).

The Weierstrass canonical product form of the Gamma function was given

by Weierstrass(1815-1897) as in the following form:

$$\Gamma(z)^{-1} = ze^{\gamma z}\prod_{k=1}^{\infty}\left(1+\frac{z}{k}\right)e^{-\frac{z}{k}} \tag{1.6}$$

where γ is the Euler-Mascheroni's constant defined by

$$\begin{aligned} \gamma &= \lim_{n\to\infty}\left(1+\frac{1}{2}+\ldots+\frac{1}{n}-\log n\right) \\ &\cong 0.577215664\ldots. \end{aligned} \tag{1.7}$$

In fact, there have been several methods to verify the duplication formula (1.5). In this note we are aiming at introducing some of them and also providing another method of proof of (1.5).

2. Some methods of verification of (1.5)

We first introduce a proof of (1.5) which was appeared in Rainville [4, p. 22-24]. The Pochhammer symbol $(\alpha)_n$ is defined by, α a complex number,

$$(\alpha)_n = \begin{cases} \alpha(\alpha+1)\ldots(\alpha+n-1) & \text{if n=1, 2, 3, } \ldots, \\ 1 & \text{if n=0,} \end{cases} \tag{2.1}$$

which is often called the generalized factorial since $(1)_n = n!$ and can be expressed as in the following

$$(\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)} \tag{2.2}$$

by using the fundamental function relation (1.3) of the Gamma function.

Rainville showed the following formula

$$(\alpha)_{2n} = 2^{2n}\left(\frac{\alpha}{2}\right)_n\left(\frac{\alpha+1}{2}\right)_n \tag{2.3}$$

by grouping alternate factors in the product

$$(\alpha)_{2n} = \alpha(\alpha+1)(\alpha+2)(\alpha+3) \quad (\alpha+2n-1).$$

Gauss also studied the Gamma function by using the following form as the definition

$$\Gamma(z) = \lim_{n\to\infty}\frac{(n-1)!n^z}{z(z+1)(z+2)\dots(z+n-1)} \tag{2.4}$$

which can now be written in the form

$$\Gamma(z) = \lim_{n\to\infty}\frac{(n-1)!n^z}{(z)_n} \tag{2.5}$$

Equation (2.5) with the aid of (2.2) yields

$$\lim_{n\to\infty}\frac{(n-1)!n^z}{\Gamma(z+n)} = 1, \tag{2.6}$$

n being an integral and z not a negative integer

Finally setting $\alpha = 2z$ in (2.3) and using (2.6) with $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ in (1.3) reduces readily to the formula (1.5).

Another proof is obtained by using (1.4) (see [1]). In (1.4) putting $p = q = x$ we obtain

$$\frac{\Gamma(x)\Gamma(x)}{2\Gamma(2x)} = 2^{1-2x}\int_0^{\frac{\pi}{2}} \sin^{2x-1} 2\theta d\theta. \tag{2 7}$$

Next we make the variable change $\phi = 2\theta$, which yields

$$\frac{\Gamma(x)\Gamma(x)}{2\Gamma(2x)} = \frac{2^{1-2x}\Gamma(\frac{1}{2})\Gamma(x)}{2\Gamma(x+\frac{1}{2})} \tag{2.8}$$

which implies (1.5).

Choi et al. [2] proved Gauss multiplication formula

$$\prod_{k=0}^{m-1} \Gamma\left(a + \frac{k}{m}\right) = (2\pi)^{\frac{1}{2}(m-1)} m^{\frac{1}{2}-ma}\Gamma(ma) \tag{2.9}$$

by observing

$$m^2\zeta(s, ma) = \sum_{k=0}^{m-1} \zeta\left(s, a + \frac{k}{m}\right)$$

and using the following formula

$$\Gamma(a) = \sqrt{2\pi}e^{\zeta'(0,a)}, \tag{2 10}$$

which can be deduced from the Hermite formula for the generalized zeta function $\zeta(s,a)$ (see Whittaker and Watson [5, pp. 269-271]), or by applying Bohr-Mollerup theorem for the Gamma function (see Choi and Nam [2]). In fact, the special case $m = 2$ of (2.9) reduces immediately to our desired formula (1.5).

In Temme [5, p. 46], the proof of (1.5) follows from

$$B(z,z) = \int_0^1 [t(1-t)]^{z-1}\,dt = 2\int_0^{\frac{1}{2}} [t(1-t)]^{z-1}\,dt.$$

Setting s=4t(1-t), we see that

$$B(z,z) = 2^{1-2z} B\left(z, \frac{1}{2}\right),$$

which is equivalent to (1.5).

3. Another proof of (1.5)

We give another proof of (1.5) by using Euler's summation formula and a simple identity involved in the logarithmic derivative of the Gamma function.

For our propose we introduce Euler summation formula which will be here referred to as a lemma to make our reasoning consistent.

LEMMA 1. *If $f'(x)$ is continuous for $x \geq 0$,*

$$\sum_{k=0}^{n} f(k) = \int_0^n f(x)dx + \frac{1}{2}f(0) + \frac{1}{2}f(n) + \int_0^n P(x)f'(x)dx, \tag{3.1}$$

where $P(x) = x - [x] - \frac{1}{2}$.

We also give a simple identity which can easily be deduced from the Weierstrass form of the Gamma function (1.6). Note that all log's in this note denote the principal branch of the logarithm function

LEMMA 2.

$$\frac{2\Gamma'(2z)}{\Gamma(2z)} - \frac{\Gamma'(z)}{\Gamma(z)} - \frac{\Gamma'(z+\frac{1}{2})}{\Gamma(z+\frac{1}{2})} = 2\log 2. \tag{3.2}$$

Proof. By taking logarithms of each member of (1 6), we obtain

$$\log\Gamma(z) = -\log z - \gamma z - \sum_{n=1}^{\infty}\left[\log\left(1+\frac{z}{n}\right) - \frac{z}{n}\right] \tag{3.3}$$

Term-by-term differentiation of the members of (13) yields

$$\frac{\Gamma'(z)}{\Gamma(z)} = -\frac{1}{z} - \gamma - \sum_{n=1}^{\infty}\left(\frac{1}{z+n} - \frac{1}{n}\right) \tag{3.4}$$

so that

$$\begin{aligned} \frac{2\Gamma'(2z)}{\Gamma(2z)} - \frac{\Gamma'(z)}{\Gamma(z)} - \frac{\Gamma'(z+\frac{1}{2})}{\Gamma(z+\frac{1}{2})} &= 2\lim_{n\to\infty}\sum_{k=n+1}^{2n}\frac{1}{2z+k} \\ &= 2\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{2z+n+k}. \end{aligned} \tag{3.5}$$

Put $f(x) = \frac{1}{2z+n+x}$ in Lemma 1, for fixed z take positive integer n such that $|z| < \frac{n}{2}$. Then $f'(x) = -\frac{1}{(2z+n+x)^2}$ is continuous for $x \geq 0$. Thus

(3.6)

$$\frac{1}{2z+n} + \sum_{k=1}^{n} \frac{1}{2z+n+k} = \sum_{k=0}^{n} \frac{1}{2z+n+k}$$

$$= \int_0^n \frac{1}{2z+n+x}\,dx + \frac{1}{2}\frac{1}{2z+n} + \frac{1}{2}\frac{1}{2z+2n} + \int_0^n P(x)\frac{-1}{(2z+n+x)^2}\,dx$$

$$= [\log(2z+n+x)]_0^n + \frac{1}{2}\frac{1}{2z+n} + \frac{1}{2}\frac{1}{2z+2n} + \int_0^n P(x)\frac{-1}{(2z+n+x)^2}\,dx.$$

Taking limit as $n \to \infty$ on each side of (3.6), we get

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{2z+n+k} = \log 2 - \lim_{n\to\infty} \int_0^n \frac{P(x)}{(2z+n+x)^2}\,dx. \tag{3.7}$$

Now

$$\begin{aligned} &\left|\int_0^n \frac{P(x)}{(2z+n+x)^2}\,dx\right| \\ &\leq \int_0^n \frac{|P(x)|}{|2z+n+x|^2}\,dx \leq \int_0^n \frac{1}{|2z+n+x|^2}\,dx \\ &\leq \int_0^n \frac{1}{(n+x)-2|z|^2}\,dx = -\frac{1}{2n-2|z|} + \frac{1}{n-2|z|}. \end{aligned} \tag{3.8}$$

We therefore have

$$\lim_{n\to\infty} \int_0^n \frac{P(x)}{(2z+n+x)^2}\,dx = 0. \tag{3.9}$$

Hence

$$\frac{2\Gamma'(2z)}{\Gamma(2z)} - \frac{\Gamma'(z)}{\Gamma(z)} - \frac{\Gamma'(z+\frac{1}{2})}{\Gamma(z+\frac{1}{2})} = 2\log 2. \tag{3.10}$$

Integrating each side of Lemma 2 from $t = \frac{1}{2}$ to z, we get

$$2\int_{\frac{1}{2}}^{z} \frac{\Gamma'(2t)}{\Gamma(2t)}\,dt - \int_{\frac{1}{2}}^{z} \frac{\Gamma'(t)}{\Gamma(t)}\,dt - \int_{\frac{1}{2}}^{z} \frac{\Gamma'(t+\frac{1}{2})}{\Gamma(t+\frac{1}{2})}\,dt = \int_{\frac{1}{2}}^{z} 2\log 2\,dt \tag{3.11}$$

so that

$$\log \frac{\sqrt{\pi}\Gamma(2z)}{\Gamma(z)\Gamma(z+\frac{1}{2})} = \log 2^{2z-1}. \tag{3.12}$$

Taking exponentials on each side of the resulting equation leads to the desired Legendre duplication formula for Gamma function,

$$\sqrt{\pi}\Gamma(2z) = 2^{2z-1}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right).$$

This completes the proof

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