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# YANG-MILLS CONNECTION IN THE ORTHONORMAL FRAME BUNDLE OVER A RIEMANNIAN SYMMETRIC SPACE

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Abstract The main result is that the connection form in the orthonormal frame bundle by the Levi-Civita connection over a Riemannian symmetric space is a Yang-Mills connection

## 1. Introduction and main result

Let $P(M,G)$ be a $G-$principal fibre bundle and $\{U,V,W,\cdots\}$ an open covering of $M$. Here $U,V,W,\cdots$ are locally trivial neighbourhoods of $P$. If the family $A=\{A_U,A_V,A_W,\cdots\}$ of the Lie algebra $\mathfrak{g}$-valued 1-forms satisfies the conditions

$$(1.1)\qquad (A_V)_x=(L_{f^{-1}(x)})_*f_{*x}+Ad(f^{-1}(x))(A_u)_X,\ x\in U\cap V,$$

where $f=f_{UV}=f_{VU}^{-1}$ are transition functions, then $A$ is called a *connection* (cf. [1, p.74]) of $P$. We denote by $\mathcal{A}_P$, the affine space of all connections over $P$. If $A$ $(\in\mathcal{A}_P)$ is a critical point of Yang-Mills functional (cf.[1, p.107])

$$(1.2)\qquad \mathcal{YM}:\mathcal{A}_P\ni A\longmapsto 1/2\int_M|F(A)|^2dv\in R,$$

then $A$ is called a *Yang-Mills connection*

To find Yang-Mills connections in a principal fibre bundle $P$ is important. It is well known that, in orthonormal frame bundle $O(M)$ of a Riemannian manifold $(M,g)$, the connection form defined by the Levi-Civita connection becomes a connection of principal fibre bundle $O(M)$. Our problem is

"*Does the above connection of principal fibre bundle $O(M)$ over $(M,g)$ become a Yang-Mills connection?*"

Recently, the second author([2]) got the following Theorem.

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THEOREM. *Let $g$ be a left invariant Riemannian metric on $SU(2)$. Then the connection form in the orthonormal frame bundle defined by the Levi-Civita connection of $g$ becomes a Yang-Mills connection if and only if $g$ is bi-invariant.*

In this paper, we obtain the following

MAIN THEOREM. *The connection form in the orthonormal frame bundle defined by the Levi-Civita connection over Riemannian symmetric space becomes a Yang-Mills connection.*

## 2. Proof of the main theorem

Let $(G/H, g)$ be a compact Riemannian symmetric space with the canonical decomposition of the Lie algebra $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$. We take a neighbourhood $U$ in $G$ and a subset $N$ (resp.$V$) of $G$ (resp.$\mathfrak{m}$) in such a way that

$(a)$ $N = U \cap exp(\mathfrak{m})$,

$(b)$ $exp : V \longrightarrow N$ and $\pi : N \longrightarrow \pi(N)$ are diffeomorphisms, where $exp(X)$, $X \in \mathfrak{g}$, is an element of 1-parameter subgroup $exp(tX)$ at $t = 1$ and $\pi$ is the restriction of the canonical projection $\pi : G \longrightarrow G/H$ to $N$. The subspace $\mathfrak{m}$ of $\mathfrak{g}$ is identified with $T_o(G/H)$, where $o := H$. We put $< \cdot,\cdot >_{\mathfrak{m}\times\mathfrak{m}} := g_o$, where $g$ is the Riemannian metric in the Riemannian symmetric space $(G/H, g)$, and define an inner product $< \cdot,\cdot >$ on $\mathfrak{g}$ by

$$(2.1) \qquad < X, Y > := \begin{cases} 0 & \text{when } X \in \mathfrak{m} \text{ and } Y \in \mathfrak{h}, \\ < X, Y >_{\mathfrak{m}\times\mathfrak{m}} & \text{when } X, Y \in \mathfrak{m}. \end{cases}$$

Let $\{e_i\}_{i=1}^{n}$ (resp. $\{e_\alpha\}_{\alpha=n+1}^{m}$) be an orthonormal basis of $\mathfrak{m}$ (resp.$\mathfrak{h}$) with respect to $< \cdot,\cdot >$. We put $(X_\alpha)_g := (dL_g)e_\alpha$ and $(X_i)_g := (dL_g)(e_i)$, $g \in G$. Here $dL_g$ is the differential of the translation $L_g : G \ni x \longmapsto gx \in G$. For $X \in V$, we define vector fields $E_i$ on $\pi(exp(V))$ by $E_i(\pi(expX)) := (d\tau_{expX})(d\pi)(e_i)$, $(i = 1, 2, \cdots, n)$. Let $\mu$ be a cross section over $\pi(expV)$, i.e.,

$$\mu : \pi(expV) \ni \pi(expX) \longmapsto expX \in expV = N.$$

We put on $\pi(expV)$

$$d\mu(E_i) = X_i + \sum_{\alpha=n+1}^{m} \xi_i^\alpha X_\alpha \tag{2.2}$$

Let $\{W^A\}_{A=1}^m$ (resp.$\{\theta^i\}_{i=1}^n$) be the (locally defined) dual frame of $\{X_A\}_{A=1}^m$ (resp.$\{E_i\}_{i=1}^n$). Then $\mu^*\omega^i = \theta^i$, and $[X_B, X_A]$ is uniquely expressed as $[X_B, X_A] = \sum_{D=1}^m C_{BA}^D X_D$. Since $(G/H, g)$ is a Riemannian symmetric space, we get

$$\begin{cases} C_{\beta\alpha}^i = 0, \quad C_{\alpha i}^\beta = 0, \quad C_{\alpha i}^j + C_{\alpha j}^i = 0, \\ C_{jk}^i = 0, \quad C_{ij}^\alpha + C_{\alpha j}^i = 0, \quad C_{\beta\gamma}^\alpha + C_{\alpha\gamma}^\beta = 0 \end{cases} \tag{2.3}$$

The connection form $\omega = (\theta_j^i)$ and the curvature form $\Omega = (\Theta_j^i)$ with respect to the frame $\{E_i\}_{i=1}^n$, on $N = expV$, in the orthonormal frame bundle over Riemannian symmetric space $(G/H, g)$ are given as follows

$$\theta_j^i = -\sum_{\alpha=n+1}^{m} \mu^* \, C_{j\alpha}^i \omega^\alpha, \tag{2.4}$$

$$\Theta_j^i = 1/2 \sum_{k,l=1}^{n} \sum_{\alpha=n+1}^{m} C_{j\alpha}^i \, C_{kl}^\alpha \, \theta^k \wedge \theta^l. \tag{2.5}$$

We denote $(\nabla_{E_k}\Omega)(E_j, E_i)$, $\omega(E_j)$ and $\Omega(E_j, E_i)$ by $\nabla_k \Omega_{ji}$, $\omega_j$ and $\Omega_{ji}$, respectively. The connection in (2.4) is a Yang-Mills connection (cf [1, p.107]) if and only if

$$(\delta_\omega \Omega)(E_i) = -\sum_{j=1}^{n} (\nabla_j \Omega_{ji} + [\omega_j, \Omega_{ji}]) = 0, (i = 1, 2, \cdots, n) \tag{2.6}$$

From straightforward computations, we get

$$(\sum_{k=1}^{n} [\omega_k, \Omega_{kl}])_s^i = \sum_{\gamma,\alpha} \sum_{t,k} (C_{t\alpha}^i C_{kl}^\alpha C_{s\gamma}^t \xi_k^\gamma - C_{t\gamma}^i C_{s\alpha}^t C_{kl}^\alpha \xi_k^\gamma), \tag{2.7}$$

$$(2.8)\qquad (\sum_{k=1}^{n} \nabla_k \Omega_{kl})^i_s = \sum_{\gamma,\alpha} \sum_{t,k} (C^i_{s\alpha} C^\alpha_{tl} C^t_{k\gamma} \xi^\gamma_k + C^i_{s\alpha} C^\alpha_{kt} \xi^\gamma_k),$$

where $(B)^i_s$ is the $(i,s)$-component of the Lie algebra $\mathfrak{o}(n)$ of orthonormal group $O(n)$. Since the curvature tensor field is parallel, we have from (2.2) and (2.4)

$$(2.9)\qquad \begin{aligned} \sum_{\beta,\alpha} \sum_{s} (C^i_{j\alpha} C^\alpha_{sl} C^s_{k\beta} \xi^\beta_h + \ & C^i_{j\alpha} C^\alpha_{ks} C^s_{l\beta} \xi^\beta_h \\ & + C^i_{s\alpha} C^\alpha_{kl} C^s_{j\beta} \xi^\beta_h - C^s_{j\alpha} C^\alpha_{kl} C^i_{s\beta} \xi^\beta_h) = 0, \end{aligned}$$

From (2.7), (2.8) and (2.9), we obtain

$$(2.10)\qquad (\delta_\omega \Omega)(E_i) = 0 \ (i = 1, 2, \cdots, n).$$

Thus the proof of the main theorem is completed.

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