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SOME REMARKS ON THE SPECTRUM OF THE LAPLACE-BELTRAMI OPERATOR

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ABSTRACT We study some spectral properties of the Laplace-Beltrami operator on compact Riemannian manifolds.

1. Introduction

Let (M, g) be a compact manifold of dimension m with metric tensor g. Let $\Delta^p = d\delta + \delta d$ be the Laplace-Beltrami operator acting on the space of smooth p-forms. Then we have the spectrum of Δ^p for each $0 \leq p \leq n$

$$Spec^p(M, g) := \{0 \leq \lambda_{1,p} \leq \lambda_{2,p} \cdots \uparrow +\infty\},$$

where each eigenvalue is repeated according to its multiplicity. Many authors (e.g.,[1,2,3,5,6,8]) have studied the relationship between the spectrum of M and the geometry of M. In this paper we shall prove ,

THEOREM A *Let (M, g) and (M', g') be compact Einstein manifolds with $Spec^p(M, g) = Spec^p(M', g')$ for an arbitrary fixed $p \geq 0$ (which implies dimM=dimM'=m). If $(m,p) \notin \{(15,1), (15,2), (16,2), (15,13), (15,14), (16,14)\}$ and $m(m-1) - 6p(m-p) \neq 0$, then (M, g) is of constant sectional curvature c if and only if (M', g') is of constant sectional curvature c'=c.*

COROLLARY *Let (M, g) be an m-dimensional compact Einstein manifold and (S^m, can) the standard Euclidean sphere. If $Spec^p(M, g) = Spec^p(S^m, can)$, then, for those pairs (m, p) listed in Theorem A, (M, g) is isometric with (S^m, can).*

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THEOREM B. *Let (M,g) and (M',g') be a compact Riemannian manifold with $Spec^p(M,g) = Spec^p(M',g')$ and $(m,p) \in \{(3,0),(3,1),(15,1),(3,2),(15,2),(16,2),(15,13),(15,14),(16,14)\}$. Then (M,g) is Einsteinian if and only if (M',g') is Einsteinian.*

2. Preliminaries and proofs

For each $p \leq m(=\dim M)$ the Minakshisundaram-Pleijel-Gaffney asymptotic expansion for $Spec^p(M,g)$ is given by

(2.1)
$$\sum_{\alpha=0}^{\infty} exp(-\lambda_{\alpha,p}t) = (4\pi t)^{-\frac{m}{2}}[a_{0,p} + ta_{1,p} + \cdots + t^N a_{N,p}] + o(t^{N-\frac{m}{2}+1}) \quad \text{as} \quad t \downarrow 0,$$

where $a_{0,p}, a_{1,p}, a_{2,p}, \cdots$ are numbers which is expressed by (see cf. [2,3])

$$a_{0,p} = \binom{m}{p} \int_M dM, \tag{2.2}$$

$$a_{1,p} = \frac{1}{6}\left[\binom{m}{p} - 6\binom{m-2}{p-1}\right] \int_M \sigma dM, \tag{2.3}$$

where σ denotes the scalar curvature of M, and for $p \notin \{0,1,2,3,m-1,m\}$

$$a_{2,p} = \alpha \int_M [P_1|C|^2 + \frac{m-3}{m-2}P_2|E|^2 + \frac{(m-2)(m-3)}{m(m-1)}P_3\sigma^2]\, dM, \tag{2.4}$$

where C, E are the Weyl conformal curvature tensor field and the Einstein tensor field respectively on M, dM denotes the volume element of M and

$$P_1 := P_1(m,p) = 2m^4 - (30p+12)m^3 + (210p^2 - 30p + 22)m^2 - (360p^3 - 30p^2 + 12)m + 180p^4,$$

$$P_2 := P_2(m,p) = -2m^4 + (180p + 18)m^3 - (900p^2 + 120p + 40)m^2 + (1440p^3 + 120p^2 + 240p + 24)m - 720p^4 - 240p^2,$$

$$P_3 := P_3(m,p) = 5m^4 - (60p + 12)m^3 + (240p^2 + 60p + 13)m^2 - (360p^3 + 60p^2 + 60p + 6)m + 180p^4 + 60p^2,$$

$$\alpha := \frac{\binom{m-4}{p-2}}{360p(p-1)(m-p)(m-p-1)}.$$

For $p \in \{0, 1, 2, 3, m-1, m\}$, the formula (2.4) is transformed into the following form ;

(2.5)
$$a_{2,p} = \beta \int_M [Q_1|C|^2 + \frac{m-3}{m-2}Q_2|E|^2 + \frac{(m-2)(m-3)}{m(m-1)}Q_3\sigma^2]\, dM,$$

where for $i = 1, 2, 3$

(i) if $p = 0$, then

$$\beta = \frac{1}{360}, \quad Q_i = Q_i(m) = \frac{P_i(m,0)}{m(m-1)(m-2)(m-3)},$$

(ii) if $p = 1$, then

$$\beta = \frac{1}{360}, \quad Q_i = Q_i(m) = \frac{P_i(m,1)}{(m-1)(m-2)(m-3)},$$

(iii) if $p = 2$, then

$$\beta = \frac{1}{2 \times 360}, \quad Q_i = Q_i(m) = \frac{P_i(m,2)}{(m-2)(m-3)},$$

(iv) if $p = 3$, then

$$\beta = \frac{1}{6 \times 360}, \quad Q_i = Q_i(m) = \frac{P_i(m,3)}{m-3},$$

(v) if $p = m - 1$, then

$$\beta = \frac{1}{360}, \quad Q_i = Q_i(m) = \frac{P_i(m,m-1)}{(m-1)(m-2)(m-3)},$$

(vi) if $p = m$, then

$$\beta = \frac{1}{360}, \quad Q_i = Q_i(m) = \frac{P_i(m,m)}{m(m-1)(m-2)(m-3)}.$$

REMARK 1. (i) The sign of the coefficients of $|C|^2, |E|^2$ and σ^2 in the formula (2.5) are respectively determined by the polynomials P_1, P_2 and P_3.
(ii) Assume that $Spec^p(M,g) = Spec^p(M',g')$. Then $\dim M = \dim M'$ is derived from (2.1)

From now on we shall write (2.4) and (2.5) in the following form ;

$$\text{(2.6)} \quad a_{2,p} = \gamma \int_M \left[R_1|C|^2 + \frac{m-3}{m-2}R_2|E|^2 + \frac{(m-2)(m-3)}{m(m-1)}R_3\sigma^2\right] dM,$$

where γ is either α or β, and R_i is either P_i or Q_i $(i = 1,2,3)$.

REMARK 2. (i) The equation $\binom{m}{p} - 6\binom{m-2}{p-1} = 0$ if and only if $m(m-1) - 6p(m-p) = 0$ if and only if $u^2 - 12v^2 = 1$, where $m = u - 1$, $p = \frac{u-1}{2} \pm v$. The least solutions are $(u,v) = (7,2), (97,28), (1351,390), \cdots$, which give $(m,p) = (6,1), (6,5), (96,20),$

$(96, 76)\cdots$.
(ii) The polynomial R_1 has the only solutions $(m,p) = (15,1), (15,2), (16,2), (15,13), (15,14), (16,14)$ (cf.[4])

Proof of Theorem A Since M and M' ($\dim M = \dim M' = m$) are Einstein manifolds, $E = 0 = E'$, and σ, σ' are constants. By Remark 2(i) and (2.3), we have $\sigma = \sigma'$. (2.6) with $Spec^p(M,g) = Spec^p(M',g')$ yields

$$\int_M R_1|C|^2 dM = \int_{M'} R_1|C'|^2 dM'$$

But for $(m,p) \notin \{(15,1), (15,2), (16,2), (15,13), (15,14), (16,14)\}$, $R_1 \neq 0$ (Remark 2(ii)). Hence $C = 0$ if and only if $C' = 0$. In particular, for $m = 3$ i.e., $C = 0 = C'$ the statement still holds. Q.E.D.

Proof of Theorem B. Assume that (M,g) is Einsteinian, i.e., $E = 0$. Then, (2.6) implies that

(2.6)
$$\int_{M'} [\frac{m-3}{m-2} R_2 |E'|^2 + \frac{(m-2)(m-3)}{m(m-1)} R_3 \sigma'^2]\, dM' = \int_M \frac{(m-2)(m-3)}{m(m-1)} R_3 \sigma^2\, dM,$$

because, for $(m,p) \in \{(3,0), (3,1), (15,1), (3,2), (15,2), (16,2), (15,13), (15,14), (16,14)\}$, $R_1 = 0$ or $C = 0$. It follows from (2.3) and σ = constant that

$$\int_{M'} \sigma'^2\, dM' \geq \int_M \sigma^2\, dM \tag{2.7}$$

Moreover, for $(m,p) \in \{(3,0), (3,1), (15,1), (3,2), (15,2), (16,2), (15,13), (15,14), (16,14)\}$,

$$R_2 > 0 \quad \text{and} \quad R_3 > 0. \tag{2.8}$$

From (2.6) $\sim$ (2.8), we conclude that $|E'|^2 = 0$, which implies that (M',g') is Einsteinian. Q.E.D

REMARK 3. Theorem A is an improvement of Theorem 3.2 in [3]. For $p = 0$ this theorem was studied in [5]. Theorem B for $p = 1, 2$ was also studied in [7] and [8] respectively.

References

[1] T. H. Kang and J S. Pak, *Some remarks for the spectrum of the p-Laplacian on Sasakian manifolds*, J Korean Math.Soc **32(2)** (1995), 341–350.

[2] V. K. Patodi, *Curvature and the fundamental solution of the heat operator*, J. Indian Math. Soc. **34** (1970).

[3] M. Puta and A. Torok, *On the spectrum of the Laplacian on p-forms*, An. Univ. Timisoara seria st. matematice **XXV**, 67–73.

[4] ———, *The spectrum of the p- Laplacian on Kahler manifold*, Rendiconti di Matematica **11** (1991), Serie VII, Roma, 257–271.

[5] T. Sakai, *On eigenvalues of Laplacian and curvature of Riemannian manifold*, Tôhoku Math J. **23**, 589–603

[6] S Tanno, *Eingenvalues of the Laplacian of Riemannian manifolds*, Tôhoku Math. J. **25**, 391–403.

[7] ———, *The spectrum of the Laplacian for 1-forms*, Proc. of A M.S. **45**, 125–129

[8] Gr. Tsagas, *On the spectrum of the Laplace operator for the exterior 2-forms*, Tensor, N S. **33**, 94–96

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