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# CERTAIN DISCRIMINATIONS OF PRIME ENDOMORPHISM AND PRIME MATRIX

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ABSTRACT In this paper, for a commutative ring $R$ with an identity, considering the endomorphism ring $End_R(M)$ of left $R-$module $_RM$ which is *(quasi-)injective* or *(quasi-)projective*, some discriminations of prime endomorphism were found as follows: each epimorphism with the *irreducible*(or *simple*) kernel on a *(quasi-)injective* module and each monomorphism with maximal image on a *(quasi-)projective* module are prime It was shown that for a field $F$, any given square matrix in $Mat_{n\times n}(F)$ with maximal image and *irreducible* kernel is a prime matrix, furthermore, any given matrix in $Mat_{n\times n}(F)$ for any field $F$ can be factored into a product of prime matrices.

## 1. Introduction

Let $R$ be a commutative ring with an identity and let $R^n$ be the direct product of $n-$copies of $R$, for any natural number $n$.

From the elementary linear algebras, it is well-known that there is an $R-$linear mapping between the set $Mat_{m\times n}(R)$ of all $m \times n-$matrices and the set $Hom_R(R^n, R^m)$ of all linear mappings from $R^n$ into $R^m$, where $n$, $m \in N$ are any natural numbers. In this paper the fact that between $Hom_R(R^n, R^m)$ and $(Mat_{m\times n}(R))^t$ there is an $R-$linear mapping too, where $t$ stands for the transpose operator is mostly used. In other words, for an element $(r_1, r_2, \cdots, r_n) \in R^n$ ,

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$$(r_1\ r_2\ \cdots\ r_n)\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{pmatrix} = (r_1, r_2, \cdots, r_n)f$$

$f$ and $(a_{ij})_{n\times m}$ act on the right side of $R^n$ for the *associated* matrix $(a_{ij})_{n\times m}$ with $f$ , denoted by $Mat(f)$ . A module ${}_RM$ is said to be $R-$*quasi-projective*( $R-$*quasi-injective*, resp.) if for each epimorphism $g : {}_RM \to {}_RN$ (for each monomorphism $f : {}_RK \to {}_RM$, resp.) and for each homomorphism $\gamma : {}_RM \to {}_RN(\gamma : {}_RK \to {}_RM$, resp.) there is an $R-$homomorphism such that $\overline{\gamma} : {}_RM \to {}_RM$ such that $\gamma = \overline{\gamma}g(\gamma : {}_RM \to {}_RM$ such that $\gamma = f\overline{\gamma}$, resp.).

Recall that $R^k$ is an $R-$(*quasi-*)*injective* and (*quasi-*)*projective* module for any natural number $k \in \mathbb{N}$. Because we are studying left $R-$modules ${}_RM$ , conveniently let the compositions of all mappings be written by the reverse order, in the order as follows:

$$gh\ : {}_RK \xrightarrow{\ g\ } {}_RM \xrightarrow{\ h\ } {}_RN$$

LEMMA 1.1. *Every monomorphism on any left $R-$(quasi-)injective module ${}_RM$ is right invertible in $End_R(M)$.*

*Proof.* In the definition of an $R-$(*quasi-*)*injective*, replacing ${}_RK$ by ${}_RM$ and $\gamma$ by the identity on ${}_RM$, the proof is established immediately.

LEMMA 1.2. *Every epimorphism on any left $R-$(quasi-)projective module ${}_RM$ is left invertible in $End_R(M)$.*

*Proof.* In the definition of an $R-$(*quasi-*)*projective*, replacing ${}_RN$ by ${}_RM$ and $\gamma$ by the identity on ${}_RM$, the proof is established immediately.

For the reason of the following definition, it will be answered partly in Remark 2.7.

An endomorphism $g$ is said to be left *retractable* in $End_R(M)$ if there is an endomorphism $g' \in End_R(M)$ such that the restriction ${}^{g'g}\mid_{Img}$ of the composition $g'g$ of $g'$ and $g$ to the image $Img$ of $g$ is the identity of the image of $g$, that is, ${}^{g'g}\mid_{Img} = 1_{Img}$ the identity endomorphism on $Img \leq M$.

DEFINITION 1.3. For a non-unit endomorphism $p$ of the endomorphism ring $S = End_R(M)$ , $p$ is said to be *prime* if $p = fg$ for $f, g \in S$ , then $f$ is right invertible or $g$ is left *retractable* in $End_R(M)$ .

Two endomorphisms $f, g \in End_R(M)$ are said to be *similar* if $Imf = Img \leq M$ . This definition of the *similarity* of two $f, g \in End_R(M)$(with an $n-$dimension vector space ${}_FM$ over a field $F = R$ having a fixed basis) is not the same as the *similarity* of two *associated* matrices $Mat(f), Mat(g)$ in $Mat_{n\times n}(R)$ with $f$, $g$ by $Mat : End_R(M) \to Mat_{n\times n}(R)$, i.e., not the same as the *similarity* of matrices in many general Linear Algebra books. Two endomorphisms $f, g \in End_R(M)$ are said to be *cosimilar* if $\ker f = \ker g \leq M$ .

## 2. Results

Any commutative ring $R$ with an identity and ${}_RR^k$ are (*quasi-*)*injective* (*quasi-*)*projective* module for any natural number $n$. By Lemmas 1.1 and 1.2, the following Theorems 2.1 and 2.2 are obtained easily.

THEOREM 2.1. *All non-unit epimorphisms on any left $R-$(quasi-)projective module ${}_RM$ are prime .*

THEOREM 2 2. *All non-unit monomorphisms on any left $R-$(quasi-)injective module ${}_RM$ are prime .*

A submodule $N \leq M$ is said to be *irreducible* (*simple*) if $N$ has no non-zero submodule.

PROPOSITION 2.3. *For a left (quasi-)projective module ${}_RM$, if a monomorphism $f$ in $End_R(M)$ has the maximal image $Imf \lneq M$ , then $f$ is prime.*

*Proof.* Suppose that $f = gh$ for some endomorphisms $g, h \in End_R(M)$ . Then the maximal submodule $Imf = Imgh \leq Imh$ implies that $Imh = M$ or $Imf = Imh$ . If $Imh = M$, then $h$ is left invertible in $End_R(M)$ since $M$ is (*quasi-*)*projective*. Hence $h$ is left *retractable.* If $Imf = Imh$, then $h = sf$ and $f = th$ for some $s, t \in End_R(M)$ since $M$ is (*quasi-*)*projective.* Thus $f = gh = gsf$ and $(1_M - gs)f = 0$ follow, where $1_M$ denotes the identity mapping on $M$. Hence $Im(1_M - gs)f =$

0 and $Im(1_M - gs) \leq \ker f = 0$ implies that $Im(1_M - gs) = 0$ and $1_M - gs = 0$. Hence $g$ is right invertible in $End_R(M)$. Therefore $f$ is prime. □

Let a *prime* monomorphism denote a monomorphism with the maximal image on a *(quasi-)projective* module.

PROPOSITON 2.4. *For any left (quasi-)injective $R-$module $_RM$, if an epimorphism $f$ in $End_R(M)$ has the irreducible kernel* $\ker f \leq M$ *, then $f$ is prime .*

*Proof.* By the dual proof of the Proposition 2.3, it is proved.

Let a *prime* epimorphism denote an epimorphism with the *irreducible*

(*simple*) kernel on an *(quasi-)injective* module.

COROLLARY 2.5. *For an endomorphism $g$ and for any prime monomorphism $f_\alpha \in End_R(M)$ with the (quasi-)projective module $_RM$, if $Img \leq \cap_\alpha Imf_\alpha$, then $f_\alpha$ divides $g$ for each $\alpha$ .*

*Proof.* Suppose that $Imf \leq \cap_\alpha Imf_\alpha$ for some indexed $\{f_\alpha\}_\alpha$. Then the fact $Imf \leq Imf_\alpha$, for each $\alpha$ implies that $f = s_\alpha f_\alpha$ for some $s_\alpha \in End_R(M)$ and for each $\alpha$ since $_RM$ is *(quasi-)projective.*

COROLLARY 2.6. *For an endomorphism $f$ and for any prime epimorphism $f_\alpha \in End_R(M)$ with (quasi-)injective module $_RM$, if* $\ker f \geq \sum_\alpha \ker f_\alpha$*, then $f_\alpha$ divides $f$ for each $\alpha$ .*

REMARK 2.7. The definition of *prime* endomorphism *(quasi-)injective* and

*(quasi-)projective* module $_RM$ if $End_R(M)$ is commutative is the same as the definition of *irreducible* or *prime* elements of commutative rings.

Precisely, on a *(quasi-)injective* and *(quasi-)projective* module $_RM$ each *prime* endomorphism $f$ with $f = gh$ implies that $g$ is a unit in $End_R(M)$ or $h$ is left *retractable* .

For a right invertible factor $g$ of $f$, there is an $s \in End_R(M)$ such that $gs = 1_M$. To show that $sg = 1_M$, let's consider the following diagram including monomorphism $g$ and epimorphism $s$ with the condition $gs = 1_M$ :

$$
\begin{array}{ccccccccc}
 & & & & M & = & M & & \\
 & & & & {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle g} & & \\
0 & \longrightarrow & M & \xrightarrow{g} & M & \xrightarrow{s} & M & \longrightarrow & 0 \\
 & & {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle s} & & \\
 & & M & = & M & = & M & &
\end{array}
$$

then there are endomorphisms $\alpha$ , $\beta \in End_R(M)$ such that $g = \alpha s$ and $s = g\beta$ since ${}_RM$ is *(quasi-)injective (quasi-)projective* Clearly $\beta\alpha = 1_M$. And hence $sg = (g\beta)(\alpha s) = g(\beta\alpha)s = gs = 1_M$ follows Therefore $g$ is a unit. Thus if $End_R(M)$ is commutative, and if a *prime* endomorphism $f$ has a product $f = gh = hg$, then one of $g$ and $h$ is at least a unit in $End_R(M)$.

From the above Corollaries 2.5 and 2.6 it isn't told in general that $f$ has a factorization in terms of the *prime* epimorphisms or the *prime* monomorphisms. It depends on the first left endomorphism and on the last right endomorphism. In other words, if $f = sp_\alpha$(or $f = p_\alpha t$) for some *prime* epimorphism or *prime* monomorphism $p_\alpha$. Then we must try to factor out $s$(or $t$) and so on, respectively

PROPOSITION 2.8. *For a left (quasi-)injective and (quasi-)projective module ${}_RM$ , if a non-unit endomorphism $f$ has the maximal image $Imf \leq M$ and the irreducible(simple) kernel* $\ker f \leq M$. *Then $f$ is prime.*

*Proof.* Suppose that $f = gh$ with $Img \cap \ker h \neq 0$ or with $Img \cap \ker h = 0$ . Then $\ker g \lneq \ker f = g^{-1}(\ker h)$ the preimage of $\ker h$ under $g$ implies that $\ker g = 0$ from $Img \cap \ker h \neq 0$ . Or we have a case of $Img \cap \ker h = 0$ with $f = gh$. If $Imp \gneq Imh = M$, the *retractability* of $h$ follows immediately. Hence we assume that $Imh = Imp \lneq M$ is proper in $M$.

We have a monomorphism $g$ which is right invertible in $End_R(M)$ for the first case. For the case of $Img \cap \ker h = 0$ , if $\ker h \neq 0$ we have a submodule $Img \oplus \ker h \leq M$. Then $Imf \simeq Img \simeq Imh$ follows from $ker f = \ker g$ and $Imh = Imp$, where the symbol $\simeq$ denotes the isomorphic. Hence $h$ is left *retractable* on $Imh$ through the extendable isomorphisms on a left (*quasi-*)*injective* and (*quasi-*)*projective* module $_RM$ .

If $\ker h = 0$, it follows that a monomorphism $h$ (which is a unit since $_RM$ is a left (*quasi-*)*injective* and (*quasi-*)*projective* module) is left *retractable* on $Imh \leq M$ . Therefore $f$ is a *prime* endomorphism. □

## 3. Applications

Remind that the ring $R$ is assumed to be a commutative ring with an identity. Here $R^n = \prod_1^n R$ denotes the product of $\{R_i\}_{1\leq i\leq n}$ with $R_i = R$ and $R^{(n)}$ denotes the direct product of $n-$copies of $R$. Recall the linear algebra theory: there is an $R-$linear mapping between the set of all linear mappings from an $n-$dimensional vector space $_FU$ into the $m-$dimensional vector space $_FV$ and the set $Mat_{n\times m}(R) = (Mat_{m\times n}(R))^t$ of $n \times m-$matrices whose entries are in $R$ where $t$ denotes the transpose operator. For any field $F$ with identity 1, the following should be noticed:

(1) Every maximal submodule of $F^{(n)}$ is the direct product $F^{(n-1)}$ and every *irreducible*(*simple*) submodule of $F^{(n)}$ is the direct product $F^{(1)}$.

(2) The direct product $F^{(n)}$ of $n-$copies of any field $F$ is (*quasi-*)*injective* and (*quasi-*)*projective* for any $n$, moreover $F^{(n)}$ is *self-generated* and *self-cogenerated.*

(3) Hereafter we assume that each $k-$dimensional space $_FF^k$ has the standard orthogonal basis
$$\{e_i = (x_1, \cdots, x_k) \mid x_i = 1, x_j = 0, \text{for } j \neq i,\ 1 \leq i, j \leq k\ \}$$
for each natural number $k \in \mathbb{N}$ .

(4) It is important to remember that every monomorphism and every epimorphism from $F^k$ into itself $F^k$ are automorphisms for every $k \in \mathbb{N}$.

Briefly and conveniently, let's replace the *associated* linear mapping $L(A)$ by $A$ properly.

APPLICATION 3.1. *Let* $A = (a_{ij}) \in Mat_{n\times n}(F)$ *be a matrix with the maximal image* $ImA = F^{(n-1)}$ *and the irreducible*(*simple*) *kernel*

$\ker A = F^{(1)}$ . *Then A is a prime matrix. Furthermore every similar cosimilar matrix to the above matrix* $(a_{ij})_{n\times n}$ *is also prime.*

*Proof.* For any epic or monic matrix $U$, the *associated* linear mapping
$L(U) : F^n \to F^n$ the (*quasi-*)*injective* (*quasi-*)*projective* $n$–dimensional vector space $F^n$ over $F$ is an automorphism. Hence each monic matrix and each epic matrix are units. Thus the Proposition 2.8 can apply here to the matrix ring $Mat_{n\times n}(F)$. Hence we have immediately a *prime* matrix $A$ with the maximal image and the *irreducible* kernel. □

For example, let $A = (a_{ij}) \in Mat_{n\times n}(F)$ be a matrix such that

$$\begin{cases} a_{jj} = 1 \ , \text{ for } 1 \le j \le n \\ a_{1k} = a_{k1} = -1 \ , \text{ for the only one } k \ , \ 1 \le k \le n \ , \\ a_{ij} = 0 \ , \ \text{ if } i \ne j \ne k \ . \end{cases}$$

$$(a_{ij})_{n\times n} = \begin{pmatrix} 1 & 0 & 0 & \cdots & -1 & \cdots & 0 \\ 0 & 1 & 0 \ldots & \ldots & \ldots & \ldots & 0 \\ 0 & 0 & 1 \ldots & \ldots & \ldots & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ -1 \ldots & \ldots & \ldots & & 0 & 1 & \cdots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 \ldots & \ldots & & \ldots & & 0 & 1 \end{pmatrix} \text{ is a } \textit{prime} \text{ matrix}$$

For this matrix $(a_{ij})_{n\times n}$ , $A = (a_{ij})_{n\times n}$ is an endomorphism with the maximal image

$$\begin{aligned} ImA &= \{(x_1, \cdots, x_k, \cdots, x_n) \mid x_1 = -x_k, \ x_i \in F \ \} \\ &= F^{(n-1)} \end{aligned}$$

and the *irreducible*(*simple*) kernel

$$\begin{aligned} \ker A &= F^{(1)} \\ &= \{ \ (a_1, 0, \cdots, 0, a_k, 0, \cdots, 0) \mid a_1 = a_k, a_i = 0 \text{ for } i \ne k, 1 \lneq i \le n \ \}. \end{aligned}$$

Corollary 3.2. *For any matrix* $A = (a_{ij})_{n\times n} \in Mat_{n\times n}(F)$ *with a field* $F$ *and for any prime matrix* $P_\alpha = (p_{ij})_\alpha \in Mat_{n\times n}(F)$, *if*

*$ImA \leq \cap ImP_\alpha$ and if $\ker A \geq \sum_\alpha \ker P_\alpha$, then $P_\alpha$ divides $A$ for each $\alpha$.*

Recalling the item (3) above the Application 3.1, we only consider the standard orthogonal bases of all $F^k$. Then the following are obtained by the *associate Mat* preserving composition of linear mappings, that is,
$Mat(fg) = Mat(f)Mat(g)$.

APPLICATION 3.3. *For $n \in N$ and for any field $F$ with an identity 1, if $f : F^n \to F^n$ is a prime endomorphism, then the associated matrix $Mat(f)$ is a prime matrix. Clearly if a square matrix is prime, then its associated linear mapping is also a prime endomorphism.*

For two matrices $A$, $C$, we call $A$ an edge factor of $C$ if $C = AB \cdots H$ or $C = H \cdots BA$ for some matrices $B, \cdots, H$.

REMARK 3.4. For two square matrices $A, B \in Mat_{n\times n}(F)$, the following are to be read about *similar* matrices :

(1) If $A, B$ are *similar* in the sense of Linear Algebra, i.e., there is a unit matrix $N \in Mat_{n\times n}(F)$ such that $A = N^{-1}BN$. Then $A$ is *prime* if and only if $B$ is *prime*. However $A$ and $B$ need not be, in general, *similar cosimilar* in the sense of this paper.
(2) For $A, B$ as in (1) and for $C \in Mat_{n\times n}(F)$, $A$ is a factor of $C$ if and only if $B$ is a factor of $C$, however for an edge factor $A$ of $C$ , $B$ need not be an edge factor of $C$ in general.
(3) Moreover for *similar cosimilar* matrices $A, B \in Mat_{n\times n}(F)$, $A$ is *prime* if and only if $B$ is *prime*.
(4) If $A, B$ are *similar cosimilar* and $C \in Mat_{n\times n}(F)$. Then $A$ is a factor, or an edge factor of $C$ if and only if $B$ is a factor, or an edge factor of $C$, respectively.

A METHOD OF FACTORIZING A SQUARE MATRIX. *There might be lots of different ways to factorize any given square matrix $A \in Mat_{n\times n}(F)$ .*

(1) *Find prime square matrices $P_\alpha$ such that $\cap ImP_\alpha \geq ImA$ and $\ker A \geq \sum \ker P_\alpha$ .*
(2) *Select one $P_{\alpha_0}$ of the prime matrices $P_\alpha$ .*
(3) *Find a factor matrix $F_{\alpha_0}$ such that $A = F_{\alpha_0}P_{\alpha_0}$ or $A = P_{\alpha_0}F_{\alpha_0}$ .*

(4) *Do the step* (1) *for the factor matrix* $F_{\alpha_0}$.
(5) *After the steps* (1) *and* (4), *go to the steps* (1) *and* (4) .
(6) *Select those factors of* $A$ *and write them properly.*

For further applications of *prime* matrices with distinct size $n$ by $m$ for $n \neq m$, here some illustrations are given.

(1) For $n \lneq m$ and a monomorphism $f : F^n \to F^m$, let $k = m - n$ and let partitionize the *associated* matrix $Mat(f)$ by $k$ by $k$, that is, $(Mat(f)) = (F_{11} \; F_{12})$ , where $F_{11} \in Mat_{k\times k}$ and $F_{12} \in Mat_{k\times n}$. Then we have a *prime* matrix $P \in Mat_{m\times m}(F)$, precisely

$$P = \begin{pmatrix} F_{11} & F_{12} \\ D_{kk} & 0_{nn} \end{pmatrix} \quad \text{where} \quad D_{kk} = (d_{ij})_{k\times k} \text{ with}$$

$$d_{ij} = \begin{cases} 0 \text{ if } i = j = l \text{ for only one } l, \; 1 \leq l \leq k \\ \delta_{ij} \text{ elsewhere , for the Kronecker's delta } \delta_{ij} \end{cases},$$

and where $0_{nn}$ is the zero matrix. This matrix $P = \begin{pmatrix} F_{11} & F_{12} \\ D_{kk} & 0_{nn} \end{pmatrix}$ is a *prime* factor of $\begin{pmatrix} Mat(f) \\ 0_{km} \end{pmatrix}_{m\times m}$.

(2) For $n \gneq m$ and an epimorphism $f : F^n \to F^m$, let $k = n - m$ and let partitionize the *associated* matrix $Mat(f)$ by $m$ by $m$, that is $(Mat(f)) = \begin{pmatrix} F_{11} \\ F_{21} \end{pmatrix}$ , where $F_{11} \in Mat_{m\times m}$ and $F_{21} \in Mat_{k\times m}$ Then we have a *prime* matrix $P$ in $Mat_{n\times n}(F)$ such that $P = \begin{pmatrix} F_{11} & 0_{mk} \\ F_{21} & D_{kk} \end{pmatrix}$ where $D_{kk} = (d_{ij})_{k\times k}$ is as in the above (1) and where $0_{mk}$ is the zero matrix. This matrix $P$ is a *prime* factor of $( Mat(f) \;\; 0_{nk} )_{n\times n}$ .

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