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CHARACTERIZATION OF HOLOMORPHY BY GLOBAL GENERATION

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1. Introduction

For an analytic coherent sheaf $\mathcal{F}$ over a Stein space X of finite dimension, the celebrated theorem of Cartan[4,5]-Oka[15]-Sèrre[17] states

Theorem A Global sections $\Gamma(X, \mathcal{F})$ generate the stalks $\mathcal{F}_x$ at each point $x \in X$.

Theorem B. $\mathrm{H}^p(X, \mathcal{F}) = 0$ for any positive integer p.

Concerning the converse of the latter, by a theorem of Cartan[2] and Benhke-Stein[1], a Cousin-I domain in $\mathbf{C}^2$ is a domain of holomorphy. Let L be a complex Lie group We denote by $\mathcal{A}_L$ the sheaf of germes of holomorphic mappings into L. We denote by $\mathcal{E}_L^\infty$ the sheaf of germes of C^∞ mappings into L. Kajiwara-Kazama[8] generalized the above Cartan-Behnke-Stein's theorem, by proving that, a domain D in a two dimensional Stein manifold is Stein if there exists a positive dimensional complex Lie group L with $\mathrm{H}^1(D, \mathcal{A}_L) = 0$ and Kajiwara-Nishihara[10] too, by characterizing the Steinness of D by the existence of L such that Oka's principle holds in the sense of the quasi-injectivity of the canonical mapping $\mathrm{H}^1(D, \mathcal{A}_L) \to \mathrm{H}^1(D, \mathcal{E}_L^\infty)$.

In case that the dimension is larger than 2, the domain $D := \mathbf{C}^3 - \{(0,0,0)\}$ is not a domain of holomorphy but satisfies $\mathrm{H}^1(D, \mathcal{O}) = 0$ by Cartan[3], where $\mathcal{O}$ is the sheaf of germs of holomorphic functions. So, the vanishing of a cohomology of degree 1 is not sufficient to let D be a domain of holomorphy. Sèrre[17] characterized a domain D of holomorphy in $\mathbf{C}^n$ by the vanishing cohomology $\mathrm{H}^p(D, \mathcal{O}) = 0$ for $p = 1, 2, \cdots, n-1$. In other words, vanishing of a suitable cohomology

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with degree from 1 to $n-1$ characterizes the Steinness of n dimensional domains.

In infinite dimensional case, Dineen[6] proved $\mathrm{H}^1(\Omega, \mathcal{O}) = 0$ for the structure sheaf $\mathcal{O}$ over a pseudoconvex domain Ω in a $\mathbf{C}$-linear space E equipped with the finite open topology. So, pseudoconvexity implies the cohomology vanishing similarly to the finite dimensional case. Raboin[16] solved the equation $\bar{\partial} f = F$ on a Hilbert space.

Kajiwara-Shon[11] proved, however, for a pseudoconvex domain Ω in the $\mathbf{C}$-linear locally convex space E equipped with the finite open topology, an analytic subset A of Ω and any positive integer $p \leq \mathrm{codim} A - 2$, the cohomology vanishing $\mathrm{H}^p(\Omega - A, \mathcal{O}) = 0$. The complement A of the open set $\Omega - A$ with respect to the pseudoconvex domain Ω has no interior point and in case codim $= \infty$ the cohomology vanishing of all positive degree does not imply that the domain is a domain of holomorphy. Moreover, Ohgai[13] proved, for any positive integers p and q, for a pseudoconvex domain Ω in the $\mathbf{C}$-linear locally convex space E equipped with the finite open topology, for a q-convex C^∞ function φ on Ω and for a negative number c, $\mathrm{H}^p(\{x \in \Omega; \varphi(x) > c\}, \mathcal{O}) = 0$. In this case, the complement $\{x \in \Omega; \varphi(x) \leq c\}$ of the open set $\{x \in \Omega; \varphi(x) > c\}$ with respect to the open set Ω may have interior points $\in \{x \in \Omega; \varphi(x) < c\}$.

So, in infinite dimensional case, vanishing of cohomology of all positive degree of the structure sheaf $\mathcal{O}$ of the domain Ω does not assure the pseudoconvexity of the domain Ω.

Kajiwara[7] proved that a finite dimensional domain D with real 1 codimensional continuous boundary in a Stein mainfold S is Stein, if and only if there exists a positive dimensional complex Lie group L such that, for any analytic polycylinder P in S, Oka's principle holds for the analytic fiber bundle with $D \bigcap P$ as base space and with L as structure group in the sense that the canonical mapping $\iota^\infty : H^1(D \bigcap P, \mathcal{A}_L) \to H^1(D \bigcap P, \mathcal{E}_L^\infty)$ is quasi-injective.

The author[12] extended the above Kajiwara's results to the infinite dimensional case.

Concerning the converse of Theorem A, Wakabayashi[18] proved the following theorem.

THEOREM OF WAKABAYASHI. *Let X be a finite dimensional connected normal complex space satisfying the following conditions: For any coherent sheaf of ideals $\mathcal{I}$ in the structure sheaf $\mathcal{O}(X)$ of X determined by a zero-dimensional analytic set in X. $\Gamma(X,\mathcal{I})$ generates $\mathcal{I}_x$ as $\mathcal{O}(X)_x$ module at each point of $x \in X$. Then X is K-complete and identical with its K-hull. If, in addition, $\Gamma(X,\mathcal{I})$ is isomorphic as a* $\mathbf{C}$*-albegra to $\Gamma(X',\mathcal{O}(X'))$ of a reduced Stein space $(X',\mathcal{O}(X'))$, then X is a Stein space.*

In the study of coherence of analytic sheaves over infinite dimensional domains, coordinates are infinite and, so, the generators may be infinite. Moreover, intersections of infinite number of open sets have not necessarily non empty open kernels.

In this way, as a Korean-Japanese joint work in Kyushu University, Kajiwara-Kim-Kim[9] gave a theorem, which takes a middle position between Theorems A and B of dimension infinite. The author successes this joint work and continues studies on a global generators of dimension infinite.

The main purpose of the present paper is to characterize Riemann domain of holomorphy by a global generation given in Kajiwara-Kim-Kim[9] in schlicht case, extending it to non schlicht case.

2. Main theorem

For any integers m and n with $m < n$, we regard the complex m-space $\mathbf{C}^m$ as a subspace

$$\mathbf{C}^m = \{z = (z_1, z_2, \cdots, z_n) \in \mathbf{C}^n; z_{m+1} = z_{m+2} = \cdots = z_n = 0\}$$

of the superspace $\mathbf{C}^n$. Let $\iota_{m,n} : \mathbf{C}^m \to \mathbf{C}^n$ be the canonical injection. We put

$$E := \sum \mathbf{C} := \bigcup_{n \geq 1} \mathbf{C}^n$$

and denote by $\iota_n : \mathbf{C}^n \to E$ and $\pi_n : E \to \mathbf{C}^n$ the canonical inclusion and projection We induce the strongest topology in E so as each injection ι_n is continuous. Then E is the infinite dimensional locally convex space equipped with the finite open topology.

Let (Ω, φ) be a *domain over* the space E, i.e., φ be a local homeomorphism of a Hausdorff space Ω in the space E. Ω is naturally equipped with the structure of a complex manifold modelled with the space E and we can speak of holomorphic functions e.t.c. on open subsets of Ω. The domain Ω is said to be *holomorphically separable* if, for any pair of two different point x and y of Ω, there exits a holomorphic function f on Ω with $f(x) \neq f(y)$.

THEOREM. *Let E be the infinite dimensional locally convex space equipped with the finite open topology and (Ω, φ) be a holomorphically separable domain over E. Then, (Ω, φ) is a domain of holomorphy over E if and only if the following conditions are satisfied:*

Let $z^{(0)} := (z_1^{(0)}, z_2^{(0)}, \cdots, z_n^{(0)}, \cdots)$ be a point of E, g be a holomorphic function on Ω and f be a holomorphic function on Ω such that, for any point $x \in \Omega$, there exists an open neighborhood U_x of x, a holomorphic function h_x on U_x and a sequence $(h_{x,1}, h_{x,2}, \cdots, h_{x,n}, \cdots)$ of holomorphic functions $h_{x,n}$ on U_x satisfying

$$(1) \qquad f = gh_x + \sum_{i=1}^{\infty}(z_i \circ \varphi - z_i^{(0)})h_{x,i} \qquad on \qquad U_x.$$

Then, there exists a holomorphic function h on Ω and a sequence $(h_1, h_2, \cdots, h_n, \cdots)$ of holomorphic functions on Ω satisfying

$$f = gh + \sum_{i=1}^{\infty}(z_i \circ \varphi - z_i^{(0)})h_i \qquad on \qquad \Omega.$$

Proof of only if part. This is a special case of Kajiwara-Kim-Kim[9] treated only the schlicht case. So, the theorem of only if part of the present paper is also a generalization in the special case. There exists a positive integer n such that $z^{(0)} \in \mathbf{C}^n$, i.e., the ν's coordinate $z_\nu^{(0)}$ of the point $z^{(0)}$ is zero when $\nu > n$. Moreover, without loss of generality, we may assume that the point $z^{(0)}$ is the origin in E.

By induction with respect to $\nu \geq n$, we prove the following Proposition P_ν:

P_ν. There exists a holomorphic function $h^{(\nu)}$ on $\varphi^{-1}(\mathbf{C}^\nu)$ and a sequence

$(h_1^{(\nu)}, h_2^{(\nu)}, \cdots, h_i^{(\nu)}, \cdots)$ *of holomorphic functions on $\varphi^{-1}(\mathbf{C}^\nu)$ satisfying*

$$f = gh^{(\nu)} + \sum_{i=1}^{\nu} z_i \circ \varphi h_i^{(\nu)} \qquad on \qquad \varphi^{-1}(\mathbf{C}^\nu).$$

and that, for any $m = n, n+1, \cdots, \nu - 1$ each h_i^{m+1} is an extension of h_i^m to $\varphi^{-1}(\mathbf{C}^{m+1})$ for $i = 1, 2, \cdots, m$.

VALIDITY OF P_n.

We consider the open covering

$$\mathcal{U}^{(n)} := \{U_x \cap \varphi^{-1}(\mathbf{C}^n); x \in \Omega\}$$

of the n-dimensional Stein manifold $\varphi^{-1}(\mathbf{C}^n)$ and the sheaf $\mathcal{R}^{(n)}$ of relations of $n+1$ holomorphic functions $g, z_1 \circ \varphi, z_2 \circ \varphi, \cdots, z_n \circ \varphi$ of holomorphic functions on $\varphi^{-1}(\mathbf{C}^n)$. Under the assumption (1), for any x and y with $U_x \cap U_y \cap \varphi^{-1}(\mathbf{C}^n) \neq \phi$, $\{(h_y - h_x, h_{y,1} - h_{x,1}, h_{y,2} - h_{x,2}, \cdots, h_{y,n} - h_{x,n}); x, y \in \Omega\}$ is a 1-cocycle of the covering $\mathcal{U}^{(n)}$ with coefficients in the sheaf $\mathcal{R}^{(n)}$. The canonical mapping $\mathrm{H}^1(\mathcal{U}^{(n)}, \mathcal{R}^{(n)}) \cong \mathrm{Z}^1(\mathcal{U}^{(n)}, \mathcal{R}^{(n)})/\mathrm{B}^1(\mathcal{U}^{(n)}, \mathcal{R}^{(n)}) \to \mathrm{H}^1(\varphi^{-1}(\mathbf{C}^n), \mathcal{R}^{(n)})$ is injective and $\mathrm{H}^1(\varphi^{-1}(\mathbf{C}^n), \mathcal{R}^{(n)}) = 0$ by Oka[14], i.e. by Theorem B. Hence, the 1-cocycle $\{(h_y - h_x, h_{y,1} - h_{x,1}, h_{y,2} - h_{x,2}, \cdots, h_{y,n} - h_{x,n}); x, y \in \Omega\}$ is a coboundary of 1-cochain $(k_x^{(n)}, k_{1,x}^{(n)}, k_{2,x}^{(n)}, \cdots, k_{n,x}^{(n)}); x \in \Omega\}$ of the covering $\mathcal{U}^{(n)}$ with coefficients in the sheaf $\mathcal{R}^{(n)}$. We put

$$h^{(n)} := h_x^{(n)} - k_x^{(n)}, \qquad h_i^{(n)} := h_{i,x}^{(n)} - k_{i,x}^{(n)} \qquad \text{on} \qquad U_x \cap \varphi^{-1}(\mathbf{C}^n)$$

for any $x \in \Omega$. Then, $h^{(n)}, h_1^{(n)}, h_1^{(n)}, \cdots, h_n^{(n)}$ are well-defined holomorphic functions on $\varphi^{-1}(\mathbf{C}^n)$ satisfying

$$f = gh^{(n)} + \sum_{i=1}^{n} z_i \circ \varphi h_i^{(n)} \qquad \text{on} \qquad \varphi^{-1}(\mathbf{C}^n),$$

what completes the proof of the validity of P_n.

VALIDITY OF $P_\nu \to P_{\nu+1}$.

We extend, respectively, the holomorphic functions $h^{(\nu)}, h_1^{(\nu)}, h_2^{(\nu)}, \cdots, h_\nu^{(\nu)}$ on the analytic subset $\varphi^{-1}(\mathbf{C}^\nu)$ of the Stein manifold $\varphi^{-1}(\mathbf{C}^{\nu+1})$ to functions

$h^{(\nu+1)}, h_1^{(\nu+1)}, h_2^{(\nu+1)}, \cdots, h_\nu^{(\nu+1)}$ holomorphic on the ambiant Stein manifold $\varphi^{-1}(\mathbf{C}^{\nu+1})$. Since the holomorphic function $f|_{\varphi^{-1}(\mathbf{C}^{\nu+1})} - g|_{\varphi^{-1}(\mathbf{C}^{\nu+1})}h^{(\nu+1)} - \sum_{i=1}^{\nu}(z_i \circ \varphi)h_i^{(\nu+1)}$ on $\varphi^{-1}(\mathbf{C}^{\nu+1})$ vanishes on the submanifold $\varphi^{-1}(\mathbf{C}^\nu)$, i.e., when $z_{\nu+1} \circ \varphi = 0$, the meromorphic function

$$h^{(\nu+1)} := \frac{f|_{\varphi^{-1}(\mathbf{C}^{\nu+1})} - g|_{\varphi^{-1}(\mathbf{C}^{\nu+1})}h^{(\nu+1)} - \sum_{i=1}^{\nu}(z_i \circ \varphi)h_i^{(\nu+1)}}{z_{\nu+1} \circ \varphi}$$

on $\varphi^{-1}(\mathbf{C}^{\nu+1})$ has points of $z_{\nu+1} \circ \varphi = 0$ as removable singularities and is holomorphically continued to a holomorphic function on $\varphi^{-1}(\mathbf{C}^{\nu+1})$, denoted by the same symbol $h^{(\nu+1)}$. Thus, we have

$$f = gh^{(\nu+1)} + \sum_{i=1}^{\nu+1} g_i h_i^{(\nu+1)} \qquad \text{on} \qquad \varphi^{-1}(\mathbf{C}^{\nu+1}),$$

what completes the proof of $P_\nu \to P_{\nu+1}$.

Proof of the if part. Assume that (Ω, φ) were not a domain of holomorphy over E. There would exist an ideal boundary point $x^{(0)}$ of (Ω, φ), an non empty open subset U of Ω, an open neighborhood W in E of the base point $z^{(0)} := (z_1^{(0)}, z_2^{(0)}, \cdots, z_n^{(0)}, \cdots) := \varphi(x^0)$ and an open proper subset V of W such that φ maps U biholomorphically onto V, that for every holomorphic function f on Ω, the holomorphic function $f \circ (\varphi|_U)^{-1}$ on $\varphi(U) = V \subset W$ would be continued holomorphically to $W \ni \varphi(x^0)$.

There exists a positive integer n such that $z^{(0)} \in \mathbf{C}^n$. Hence the stalk $\varphi^{-1}(z^{(0)})$ is at most countable. We count points of $\varphi^{-1}(z^{(0)})$ as $\{x^{(\nu)}; \nu = 1, 2, 3, \cdots\}$. Since Ω is holomorphically separable, the points $x^{(0)}$ and $x^{(\nu)}$ are separated by a holomorphic function on Ω, there exists a holomorphic function $f_\nu(x)$ on Ω such that $f_\nu(x^{(0)}) =$

0 and $f_\nu(x^{(\nu)}) \neq 0$ for any $\nu \geq 1$. Let $\{K_\nu; \nu = 1,2,3,\cdots\}$ be a sequence of compact subsets K_ν of Ω such that $K_\nu \subset \varphi^{-1}(\mathbf{C}^\nu)$ and that K_ν is contained in the interior of $K_{\nu+1}$ in $\varphi^{-1}(\mathbf{C}^{\nu+1})$. Let $a := (a_1, a_2, \cdots, a_\nu, \cdots)$ be an element of the Hilbert space ℓ^2 of square summable sequences. We put

$$(2) \qquad g_a(x) := \sum_{\nu=1}^{\infty} \frac{1}{2^\nu} \frac{a_\nu f_\nu(x)}{1 + \sup_{y \in K_\nu} |f(y)|} \qquad \text{on} \qquad \Omega.$$

Then $g_a|_{\varphi^{-1}(\mathbf{C}^\nu)}$ converges uniformly on any compact subset of $\varphi^{-1}(\mathbf{C}^\nu)$ and g_a is a holomorphic function on Ω. Since the set S_ν of a's with $g_a(x^{(\nu)}) = 0$ is a nowhere dense closed subset of the Hilbert space ℓ^2 by the theorem of Riesz, the complement of their union $\cup_{\nu=1}^{\infty} S_\nu$ has an exterior point a in ℓ^2 by the therem of Baire For this a, we denote g_a simply by g. Then the holomorphic function g on Ω separates the point $x^{(0)}$ and other points in the stalk $\varphi^{-1}(z^{(0)})$ simultaneously, i.e., we have $g(x^{(0)}) = 0$ and $g(x^{(\nu)}) \neq 0$ for $\nu = 1,2,3,\cdots$. Let x be a point of Ω. Then, two cases may occur. Firstly, $w := (w_1^{(0)}, w_2^{(0)}, \cdots, w_n^{(0)}, \cdots) := \varphi(x) \neq z^{(0)}$. There exits a positive integer m such that $w_m^{(0)} \neq z_m^{(0)}$. Since the coordinate function $z_m \circ \varphi - z_m^{(0)}$ is continuous, there exits an open neighborhood $W(x)$ of the point x such that $z_m \circ \varphi - z_m^{(0)} \neq 0$ in $W(x)$ Then, for the holomorphic function $h_{m,x}$ on W defined by

$$h_{m,x} := \frac{1}{z_m \circ \varphi - z_m^{(0)}},$$

we have

$$1 = (z_m \circ \varphi - z_m^{(0)}) h_{m,x} \qquad \text{on} \qquad W(x).$$

In the second case that $\varphi(x) = z^{(0)}$, the point x belongs to the said $\{x^{(\nu)}; \nu = 1,2,3,\cdots\} = \varphi^{-1}(z^{(0)}) - x^{(0)}$, the holomorphic function g constructed by (2) satisfies $g(x) \neq 0 = g(x^{(0)})$. There exits an open neighborhood $W(x)$ of the point x such that $g(y) \neq 0$ for $y \in W(x)$. Then, for the holomorphic function h_x on $W(x)$ defined by

$$h_x := \frac{1}{g(y)} \qquad (y \in W(x)),$$

we have

$$1 = gh_x \qquad \text{on} \qquad W(x).$$

Thus the assumption (1) is satisfied. Then, there would exist a holomorphic function h on Ω and a sequence $\{h_i; i = 1, 2, \cdots, n, \cdots\}$ of holomorphic functions on Ω such that we would have

$$1 = gh + \sum_{i=1}^{\infty}(z_m \circ \varphi - z_m^{(0)})h_i$$

for the constant function 1 on Ω. By the assumption of the reduction to absurd, the holomorphic functions $g \circ (\varphi|_U)^{-1}, h \circ (\varphi|_U)^{-1}$ and $h_i \circ (\varphi|_U)^{-1}$ are holomorphically continued to holomorphic functions on $W \ni z^{(0)}$, which are denoted by the same symbols. We substitute $z = z^{(0)}$ in the equation

$$1 = g \circ (\varphi|_U)^{-1} h \circ (\varphi|_U)^{-1} + \sum_{i=1}^{\infty}(z_m - z_m^{(0)})h_i \circ (\varphi|_U)^{-1}$$

and get the equation $1 = 0$, what is a contradiction.

References

[1] H. Behnke und K. Stein, *Analytische Funktionen mehrerer Veränderlichen zu vorgebenen Null-und Pol-stellenflachen*, Jber. D. M. V. **47** (1937), 177–192.

[2] H. Cartan, *Le problème de Poincaré et de Cousin pour les fonctions de plusieurs variables complexes*, C. R. Paris **198** (1934), 1284–128.

[3] ———, *Sur les première problème de Cousin*, C R Paris **207** (1938), 558–560

[4] ———, *Séminaire sur les fonctions de plusieurs variables*, Ecole Normale Supérieur, Paris, 1951-1952

[5] ———, *Séminaire sur les fonctions de plusieurs variables*, Ecole Normale Supérieur, Paris, 1953-1954.

[6] S. Dineen, *Sheaves of holomorphic functions on infinite dimensional vector spaces*, Math Ann. **202** (1973), 337–345

[7] J. Kajiwara, *Equivalence of Steinness and validity of Oka's principle for domains with continous boundaries of a Stein manifold*, Mem. Fac. Sci. Kyushu Univ. **33** (1979), 83–93.

[8] J. Kajiwara and H. Kazama, *Two dimensional complex manifold with vanishing cohomology set*, Math Ann. **204** (1973), 1–12.

[9] J. Kajiwara, D K Kim and J -J. Kim, *Analytic Coherent Sheaves on Infinite Dimension*, Proc. Finite or Infinite Dim. Comp Anal **4** (1996), 225–226.

[10] J. Kajiwara und M. Nishihara, *Charactersierung der Steinschen Teilgebieten durch Okasches Prinzip in Zwei-dimensionater Mannigfasltigkeit*, Mem. Fac Sci.Kyushu **33** (1979), 71–76.

[11] J Kajiwara and K. H. Shon, *Continuation and vanishing theorem for cohomology of infinite dimensional space*, Pusan Kyŏngnam Math. J **9** (1993), 65–73.

[12] X D Li, *Characterization of holomorphy by vaslisity of Oka's principle*, Bull Hong Kong Math Soc **in Press** (1998).

[13] S Ohgai, *Cohomology vanishing and q-convexity functions on infinite dimensional domains*, Proceedings of the International Colloquium on Differential Equations VSP(Netherland Utrecht) **7** (1997), 277–282

[14] K Oka, *Sur les fonctions analytiques plusieurs variables. VII Sur quelque notion arithmétiques*, Bull. Soc. Math France **78** (1950), 1–27

[15] ———, *Sur les fonctions analytiques plusieurs variables: IX Domaines finis sans point critique intérieur*, Jap J Math **23** (1953), 1–27

[16] P Raboin, *Resolution de l'équation* $\overline{\partial} f = F$ *sur un espace de Hilbert*, Bull Soc. Math France **107** (1979), 225–240

[17] J P. Sèrre, *Quelque problème globaux relatifs aux variété de Stein*, Coll fonct. plus. var Bruxelles (1953), 57–68

[18] I. Wakabayashi, *A remark on theorem A for Stein Spaces*, J Math. Soc. Japan. **11** (1957), 489–4923.

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