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AN ECONOMICAL KATO'S DECOMPOSITION

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Abstract In this note, we give an economical Kato's decomposition for a regular operator on a Banach space.

1. Introduction

If X is a Banach space, we shall write $B(X)$ for the set of all bounded linear operators on X. For brevity, we shall write : if $T \in B(X)$, then

$$\alpha(T) = dim\, T^{-1}(0) \quad \text{and} \quad \beta(T) = dim\, X/cl\, T(X).$$

Thus, $\alpha(T)$ and $\beta(T)$ will be either a non-negative integer or ∞. We recall ([1]) that $T \in B(X)$ is called *upper semi-Fredholm* if

$$T \quad \text{has a closed range and} \quad \alpha(T) < \infty$$

and is called *lower semi-Fredholm* if

$$T \quad \text{has a closed range and} \quad \beta(T) < \infty.$$

If $T \in B(X)$ is either upper or lower semi-Fredholm, it is called *semi-Fredholm* operator.
If $T \in B(X)$ is semi-Fredholm, then by the punctured neighborhood theorem ([1],[3]), there is $\epsilon > 0$ for which $\alpha(T-\lambda)$ and $\beta(T-\lambda)$ are both constant for $0 < |\lambda| < \epsilon$. Thus we can define ([6]) the *jump*, $j(T)$, of a semi-Fredholm operator T :

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$$j(T) = \begin{cases} \alpha(T) - \alpha(T-\lambda) & \text{for } 0 < |\lambda| < \epsilon \text{ if } T \text{ is upper semi-Fredholm} \\ \beta(T) - \beta(T-\lambda) & \text{for } 0 < |\lambda| < \epsilon \text{ if } T \text{ is lower semi-Fredholm.} \end{cases}$$

Continuity of the index ensures that the jump is unambiguously defined for Fredholm operators.

2. Main Result

If $T \in B(X)$ we can introduce ([1],[2]) the *hyperrange* and the *hyperkernel* of T :

$$T^{\infty}(X) = \cap_{n=1}^{\infty} T^n(X) \quad \text{and} \quad T^{-\infty}(0) = \cup_{n=1}^{\infty} T^{-n}(0).$$

It is clear that both subspaces are invariant under any operator $S \in B(X)$ which commutes with T. It is well known ([5],[6]) that if $T \in B(X)$ is semi-Fredholm, then

$$j(T) = 0 \iff T^{-1}(0) \subseteq T^{\infty}(X) \iff T^{-\infty}(0) \subseteq T(X).$$

Then *Kato's decomposition theorem* ([4],[6]) says that if $T \in B(X)$ is semi-Fredholm, T can be decomposed as :

$$T = T_1 \oplus T_2,$$

where T_1 is nilpotent and $j(T_2) = 0$.

We recall ([1]) that $T \in B(X)$ is called *regular* if there is $T' \in B(X)$ for which

$$T = TT'T; \tag{0.1}$$

then T' is called a *generalized inverse* for T. It is familiar that we can always arrange

$$T'' = T'TT' : \tag{0.2}$$

indeed if (0.1) holds with $T' = T''$ then (0.1) and (0.2) hold with $T' = T''TT''$.

LEMMA 1. *If $T = TT'T \in B(X)$ is regular and $T^{-k}(0) \subseteq T(X)$ for some $k \in N$, then*

$$T(T')^{k+1}T^{k+1} = (T')^k T^{k+1}. \tag{1.1}$$

Proof. If T is regular with $T = TT'T$, then

$$T^{-1}(0) = (I - TT')(X) \quad \text{and} \quad T(X) = (I - TT')^{-1}(0).$$

If $k = 1$, then

$$(I - TT')(I - T'T) = 0$$

so that

$$T(T')^2 T = TT' + T'T - I,$$

and hence

$$T(T')^2 T^2 = T'T^2. \tag{1.2}$$

Further, (1.2) says that T^2 is regular with the generalized inverse $(T')^2$. Now the proceeding argument with $T = T^k (k = 2, 3, \cdots)$ gives the inductive step, which gives (1.1).

The following theorem is the main result of this note.

THEOREM 2. *If T is regular with $TT'T = T$ and $T'TT' = T'$ and is upper semi-Fredholm, then*

$$j(T) = j(T_N) + dim\, L,$$

with

$$L = T^{-k}(0) \ominus T^{-k}(0) \cap T(X)$$

and

$$N = (T')^k T^k(X) \bigoplus \left[T^{-k}(0) \ominus (\oplus_{i=0}^{k-1} T^i L)\right],$$

where k is the smallest integer such that

$$T^{-(k-1)}(0) \subseteq T(X) \quad but \quad T^{-k}(0) \not\subseteq T(X).$$

Proof. Suppose T is upper semi-Fredholm and regular. If $j(T) \neq 0$, then there is the smallest integer k such that

$$T^{-(k-1)}(0) \subseteq T(X) \quad \text{but} \quad T^{-k}(0) \nsubseteq T(X).$$

Since $T^{-k}(0)$ is finite dimensional, there is a closed subspace L such that

$$L \oplus (T^{-k}(0) \cap T(X)) = T^{-k}(0).$$

Also we have the following decomposition of $T^{-k}(0) \cap T(X)$:

$$T^{-k}(0) \cap T(X) = \overbrace{TL \oplus \cdots \oplus T^{k-1}L \oplus A_1}^{T^{-(k-1)}(0)} \bigoplus A_2,$$

where $dimL = dimT^iL$ for $i = 1, 2, \cdots, k-1$. Further, since $T^{-k}(0) + T(X)$ is complemented, we can find closed subspaces A_3 and A_4 of X for which

$$X = L \bigoplus \underbrace{\overbrace{\oplus_{i=1}^{k-1} T^i L \oplus A_1}^{T^{-(k-1)}(0)} \oplus A_2 \oplus A_3}_{T(X)} \bigoplus A_4.$$

Since T is regular, we can find a generalized inverse T' such that

$$T = TT'T \quad \text{and} \quad T'TT' = T'$$

satisfying that, via Lemma 1.

$$(T')^k T^k = A_3 \oplus A_4 \quad \text{and} \quad (T')^{k-1} T^{k-1} = L \oplus A_2 \oplus A_3 \oplus A_4.$$

The last assertion comes from the fact that

$$\begin{aligned} (T')^{k-1}T^{k-1}((T')^k T^k) &= (T')^{k-1}T^{k-1}(T')^{k-1}(T'T^k) \\ &= (T')^{k-1}T'T^k \\ &= (T')^k T^k. \end{aligned}$$

Put

$$M = \bigoplus_{i=0}^{k-1} T^i L \quad \text{and} \quad N = \bigoplus_{i=1}^{4} A_i.$$

Then we claim that

$$T(M) \subseteq M \quad \text{and} \quad T(N) \subseteq N.$$

The first inclusion is evident and the second inclusion comes from the following calculation :

$$T(A_1 \oplus A_4) \subseteq A_1 \oplus A_2$$

and

$$\begin{aligned} T(A_3 \oplus A_4) &= T((T')^k T^k)(X) \\ &= (T')^{k-1} T^k (X) \qquad \text{(by (1.1))} \\ &= (T')^{k-1} T^{k-1}(T(X)) \\ &= (T')^{k-1} T^{k-1}(T^{-(k-1)}(0) \oplus A_2 \oplus A_3) \\ &= (T')^{k-1} T^{k-1}(A_2 \oplus A_3) \\ &= A_2 \oplus A_3. \end{aligned}$$

Evidently, T_M is nilpotent of nilpotency k and

$$T^{-1}(0) = T^{k-1} L \text{ and hence } j(T_M) = dim\, L,$$

and hence $\quad j(T_N) = j(T) - dim\, L.$

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