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# INVERSE POLYNOMIALS, RAMIFICATION INDICES AND INJECTIVE ENVELOPES OF SIMPLE MODULES OVER POLYNOMIAL RINGS

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## 1. Introduction

R. M. Fossum[F] gave a precise description, for a prime ideal $\mathfrak{p}$ in a commutative Noetherian ring $R$, of the indecomposable injective $R$-module $E(R/\mathfrak{p})$, where $E(\ \ )$ is used to denote 'injective envelope'. Fossum's description involves localization of $R$ at $\mathfrak{p}$ , completion of the local ring $R_\mathfrak{p}$, use of Cohen's structure theorem for a complete local ring to express $(R_\mathfrak{p})^\wedge$ as a homomorphic image of an appropriate ring of formal power series, and then use of modules of inverse polynomials (see [F. Section II]).

However, in some circumstances, one can obtain an explicit description of $E(R/\mathfrak{p})$ without carrying out all the steps in Fossum's recipe .

For example, for $R = K[X_1, \cdots, X_n]$, the ring of polynomials over a field $K$ in indeterminates $X_1, \cdots, X_n$, and $\mathfrak{m} = (X_1, \cdots, X_n)$, it was shown by D. G. Northcott[N2] that $E(R/\mathfrak{m})$ is isomorphic to the $R$-module of inverse polynomials $K[X_1^{-1}, X_2^{-1}, \cdots, X_n^{-1}]$. From this, one can easily obtain an explicit description of the injective envelope of an arbitrary simple module over $K[X_1, \cdots, X_n]$ when $K$ is algebraically closed. In [So] and [SK], we gave

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concrete descriptions, in terms of modules of generalized fractions, of the injective envelope of an arbitrary simple module over $K[X_1, \cdots, X_n]$, where $K$ has characteristic zero but is not necessarily algebraically closed. One of the main results of this paper is to give a little different description, in terms of modules of inverse polynomials and Galois theory, of the above injective envelope. The description we give is rather different from that resulting from Fossum's structure theorem. And we shall be concerned with connections between ramification indices, residue class degrees and the above mentioned injective envelope.

Throughout the paper, we shall use $K$ to denote a field, and $A$ will denote $K[X_1, \cdots, X_n]$, the ring of polynomials over $K$ in $n$ indeterminates(where $n > 0$). Also, $\mathfrak{m}$ will denote a maximal ideal of $A$. We use $E_A(N)$ to denote the injective envelope of the $A$-module $N$.

## 2. Modules of inverse polynomials

In this section, we recall a result of D. G. Northcott[N2], and point out some easy consequences.

REMARK 2.1. Consider the $A$-module of 'inverse polynomials' $K[X_1^{-1}, \cdots, X_n^{-1}]$ described in [N2]. By [N2, Theorem 2],

$$K[X_1^{-1}, \cdots, X_n^{-1}] \cong E_A(K[X_1, \cdots, X_n]/(X_1, \cdots, X_n)).$$

Now let $a_1, \cdots, a_n \in K$. Since $K[X_1, \cdots, X_n] \cong K[X_1 - a_1, \cdots, X_n - a_n]$, so that $X_1 - a_1, \cdots, X_n - a_n$ are algebraically independent over $K$, it follows that

$$E_A(K[X_1, \cdots, X_n]/(X_1 - a_1, \cdots, X_n - a_n)) \cong K[(X_1 - a_1)^{-1}, \cdots, (X_n - a_n)^{-1}].$$

Note that the $A$-module structure on $K[(X_1 - a_1)^{-1}, \cdots, (X_n - a_n)^{-1}]$ is such that, for $r_1, \cdots, r_n \in \mathbb{N}_0$(the set of non-negative integers),

$$(X_i - a_1)((X_1 - a_1)^{-r_1}(X_2 - a_2)^{-r_2} \cdots (X_n - a_n)^{-r_n}) = \begin{cases} (X_1 - a_1)^{-r_1} \cdots (X_i - a_i)^{-r_i+1} \cdots (X_n - a_n)^{-r_n} & , r_i > 0 \\ 0 & , r_i = 0. \end{cases}$$

Thus modules of inverse polynomials enable us to describe, up to isomorphism, the injective envelope of a simple $A$-module $A/\mathfrak{m}$ for $\mathfrak{m}$ of the form $(X_1 - a_1, \cdots, X_n - a_n)$ with $a_1, \cdots, a_n \in K$. In the case in which $K$ is algebraically closed, it follows from the Nullstellensatz that this process describes, up to isomorphism, the injective envelope of every simple $A$-module.

## 3. Inverse polynomials and injective envelopes of simple modules over polynomial rings

In this section, we are going to approach our problem by considering an appropriate extension field of $K$ in which we can use the results of 2.1. We begin by setting up notation which will be in force throughout the remainder of the paper. The reader is referred to [S0, §2] for more details.

**NOTATION and TERMINOLOGY 3.1.** Throughout the paper, $\bar{K}$ will denote an algebraic closure of $K$.

If $L$ is an algebraic extension field of $K$, then $A = K[X_1, X_2, \cdots, X_n]$ is a subring of $L[X_1, \cdots, X_n]$, and the latter ring is integral over $A$. We shall say that a prime ideal $\mathfrak{q}$ of $L[X_1, \cdots, X_n]$ *lies over* $\mathfrak{m}$ if $\mathfrak{q} \cap A = \mathfrak{m}$. Observe that such a $\mathfrak{q}$ must be maximal, and also must be a minimal prime ideal of $\mathfrak{m}L[X_1, \cdots, X_n]$; hence there are only finitely many maximal ideals of $L[X_1, \cdots, X_n]$ which lie over $\mathfrak{m}$. Let $L$ be an algebraic extension field of $K$. We shall say that $\mathfrak{m}$ *splits in* $L$ if the (necessarily finitely many) maximal ideals $\mathfrak{m}_1, \cdots, \mathfrak{m}_t$ of $L[X_1, \cdots, X_n]$ which lie over $\mathfrak{m}$ are such that, for suitable $a_{ij} \in L (i = 1, \cdots, t, j = 1, \cdots, n)$,

$$\mathfrak{m}_i = (X_1 - a_{i1}, \cdots, X_n - a_{in}) \quad \text{for all} \quad i = 1, \cdots, t.$$

It is easy to see that if $L$ is an algebraic extension field of $K$ such that $\mathfrak{m}$ splits in $L$, and $L'$ is an algebraic extension field of $L$, then $\mathfrak{m}$ splits in $L'$.

LEMMA 3.2. *([S0,(2.5)]) There exists a finite field extension $L$ of $K$, with $K \subseteq L \subseteq \bar{K}$, such that $\mathfrak{m}$ splits in $L$.*

**ADDITIONAL NOTATION 3.3.** For the remainder of the paper, we shall suppose that $L$ is a finite extension field of $K$ such that $\mathfrak{m}$ splits in $L$. We shall let $\mathfrak{m}_1, \cdots, \mathfrak{m}_t$ be the maximal ideals of $L[X_1, \cdots, X_n]$ which lie over $\mathfrak{m}$, and we shall suppose that $a_{ij} \in L (i = 1, \cdots, t, j = 1, \cdots, n)$ are such that

$$\mathfrak{m}_i = (X_1 - a_{i1}, \cdots, X_n - a_{in}) \quad \text{for all} \quad i = 1, \cdots, t.$$

We shall denote $L[X_1, \cdots, X_n]$ by $B$, and we shall let $G := Gal(L : K)$ denote the Galois group of $L$ over $K$. Note that each $\sigma \in G$ induces an isomorphism of the ring $B$ which has restriction to $A$ equal to the identity and restriction to $L$ equal to $\sigma$ : we shall denote this induced isomorphism also by $\sigma$.

For each $i = 1, \cdots, t$, let $E(B/\mathfrak{m}_i) = L[(X_1 - a_{i1})^{-1}, \cdots, (X_n - a_{in})^{-1}]$ ; by 2.1, $E(B/\mathfrak{m}_i)$ is an injective envelope of the $B$-module $B/\mathfrak{m}_i$. Lastly, set $E = \bigoplus_{i=1}^{t} E(B/\mathfrak{m}_i)$, an injective $B$-module.

REMARK 3.4. Let $\sigma \in G$, and let $j \in \mathbb{N}$ (the set of positive integers) with $1 \leq j \leq t$. Then $\sigma(\mathfrak{m}_j)$ must also be a maximal ideal of $B$ which lies over $\mathfrak{m}$, and so must be $\mathfrak{m}_k$ for some $k$ with $1 \leq k \leq t$. It follows that $\sigma(a_{ji}) = a_{ki}$ for all $i = 1, \ldots, n$.

It is easy to see that $\sigma$ induces an isomorphism of $A$-modules

$$\sigma^{(j)} : E(B/\mathfrak{m}_j) \longrightarrow E(B/\mathfrak{m}_k)$$

which is such that

$$\sigma^{(j)}(l(X_1 - a_{j1})^{-r_1} \cdots (X_n - a_{jn})^{-r_n}) \\ = \sigma(l)(X_1 - a_{k1})^{-r_1} \cdots (X_n - a_{kn})^{-r_n}$$

for all $l \in L, r_1, \cdots, r_n \in \mathbb{N}_0$. It follows that $\sigma$ induces an $A$-module automorphism of $E$ which has restriction to $E(B/\mathfrak{m}_j)$ equal to $\sigma^{(j)}$ for all $j = 1, \cdots, t$. We shall denote this induced automorphism also by $\sigma$.

LEMMA 3.5. *([S0,(2.10)]) Assume that the order of $G$ is not divisible by the characteristic of $K$. Then*

$$E^G := \{e \in E | \sigma(e) = e \quad \textit{for all} \quad \sigma \in G\}$$

*is an injective $A$-module.*

THEOREM 3.6. *Assume that $L$ is a separable normal extension of $K$. The Galois group $G$ acts on transitively on $\{\mathfrak{m}_1, \cdots, \mathfrak{m}_t\}$; for each $i = 1, \cdots, t$, let $\sigma_i \in G$ be such that $\sigma_i(\mathfrak{m}_1) = \mathfrak{m}_i$, with the understanding that $\sigma_1$ is the identity. Set $a_j = a_{1j}$ for all $j = 1, \cdots, n$, so that*

$$\mathfrak{m}_1 = \sum_{i=1}^{n}(X_i - a_i)B$$

*and*

$$\mathfrak{m}_j = \sum_{i=1}^{n}(X_i - \sigma_j(a_i))B \quad \text{for all} \quad j = 2, \cdots, t.$$

*Let $K' = K(a_1, \cdots, a_n)$. Then,*

$$E^G = \{(\delta, \sigma_2^{(1)}(\delta), \cdots, \sigma_t^{(1)}(\delta)) \in E \mid \sigma \in K'[(X_1 - a_1)^{-1}, \cdots, (X_n - a_n)^{-1}]\}.$$

*Proof.* The fact that $G$ acts transitively on $\{\mathfrak{m}_1, \cdots, \mathfrak{m}_t\}$ is well known.

First, let $e = (\delta_1, \cdots, \delta_t) \in E^G$, so that $\delta_1 \in E(B/\mathfrak{m}_1)$ : we can write

$$\delta_1 = \sum_{i=1}^{w} l_i(X_1 - a_1)^{-\alpha_{i1}} \cdots (X_n - a_n)^{-\alpha_{in}},$$

where, $w \in \mathbb{N}_0, l_1, \cdots, l_w \in L\backslash\{0\}$, and $(\alpha_{i1}, \cdots, \alpha_{in})(i = 1, \cdots, w)$ are $w$ distinct elements of $\mathbb{N}_0^n$.(An empty sum is regarded as zero.) Let $\sigma \in Gal(L : K')$, a subgroup of $G$. Then $\sigma(\mathfrak{m}_1) = \mathfrak{m}_1$, and so, since $\sigma(e) = e$, we must have $\sigma^{(1)}(\delta_1) = \delta_1$, that is,

$$\sum_{i=1}^{w} \sigma(l_i)(X_1 - a_1)^{-\alpha_{il}} \cdots (X_n - a_n)^{-\alpha_{in}} = \sum_{i=1}^{w} l_i(X_1 - a_1)^{-\alpha_{i1}} \cdots (X_n - a_n)^{-\alpha_{in}}.$$

It follows that $\sigma(l_i) = l_i$ for all $i = 1, \cdots, w$. As this holds for all $\sigma \in Gal(L : K')$, and as $L$ is a finite, normal, separable extension of $K$, it follows from the Fundamental Theorem of Galois theory that $l_i \in K'$ for all $i = 1, \cdots, w$. Hence $\delta_1 \in K'[(X_1 - a_1)^{-1}, \cdots, (X_n - a_n)^{-1}]$, and, since we must have, for each $i = 2, \cdots, t$, that $\delta_i = \sigma_i^{(1)}(\delta_1)$ simply because $e = \sigma_i(e)$, we have proved that

$$E^G \subseteq \{(\delta, \sigma_2^{(1)}(\delta), \cdots, \sigma_t^{(1)}(\delta)) \in E \mid \\ \delta \in K'[(X_1 - a_1)^{-1}, \cdots, (X_n - a_n)^{-1}]\}.$$

The opposite inclusion can be proved similarly.

THEOREM 3.7. *Assume that $L$ is a separable normal extension of $K$, and the order of $G$ is not divisible by the characteristic of $K$. Then $E^G$ is an injective envelope of the simple $A$-module $A/\mathfrak{m}$.*

*Proof.* By 3.5, $E^G$ is an injective $A$-module. We shall use the description of $E^G$ obtained in 3.6 to show that it is an injective envelope of $A/\mathfrak{m}$. By 3.6, the element

$$\zeta := (1, 1, \cdots, 1) \in E = \bigoplus_{i=1}^{t} L[(X_1 - a_{i1})^{-1}, \cdots, (X_n - a_{in})^{-1}]$$

of $E$ actually belongs to $E^G$. Clearly, $\zeta \neq 0$ and $\mathfrak{m}\zeta = 0$, so that $S := A\zeta$ is a simple $A$-submodule of $E^G$ and $S \cong A/\mathfrak{m}$. Let $l \in L\backslash\{0\}$ and $\alpha_1, \cdots, \alpha_n \in \mathbb{N}_0$ with $\alpha_1 > 0$. We consider the effect of multiplication by $(X_1 - c)$, for $c \in L$, on the element

$$l(X_1 - a_1)^{-\alpha_1} \cdots (X_n - a_n)^{-\alpha_n} \in E(B/\mathfrak{m}_1) \\ = L[(X_1 - a_1)^{-1}, \cdots, (X_n - a_n)^{-1}].$$

First,

$$(X_1 - a_1)(l(X_1 - a_1)^{-\alpha_1} \cdots (X_n - a_n)^{-\alpha_n}) \\ = l(X_1 - a_1)^{-\alpha_1+1}(X_2 - a_2)^{-\alpha_2} \cdots (X_n - a_n)^{-\alpha_n} \neq 0.$$

Secondly, for $c \in L$ with $c \neq a_1$,

$$(X_1 - c)(l(X_1 - a_1)^{-\alpha_1} \cdots (X_n - a_n)^{-\alpha_n}) \\ = (X_1 - a_1 + a_1 - c)l(x_1 - a_1)^{-\alpha_1} \cdots (X_n - a_n)^{-\alpha_n} \\ = l(X_1 - a_1)^{-\alpha_1+1}(X_2 - a_2)^{-\alpha_2} \cdots (X_n - a_n)^{-\alpha_n} \\ + (a_1 - c)l(X_1 - a_1)^{-\alpha_1}(X_2 - a_2)^{-\alpha_2} \cdots (X_n - a_n)^{-\alpha_n},$$

and both terms in this last expression are nonzero. Now let $m_1$ be the minimal polynomial of $a_1$ over $K$. Bearing in mind that $L$ is a normal separable extension of $K$, it follows from ideas like those in the preceding paragraph that

$$m_1(X_1)^{\alpha_1}(l(X_1 - a_1)^{-\alpha_1} \cdots (X_n - a_n)^{-\alpha_n}) \\ = l'(X_2 - a_2)^{-\alpha_2} \cdots (X_n - a_n)^{-\alpha_n}$$

for some $l' \in L \backslash \{0\}$. Note also that

$$m_1(X_1)^{\alpha_1+1} l(X_1 - a_1)^{-\alpha_1} \cdots (X_n - a_n)^{-\alpha_n} = 0.$$

We now return to the problem of showing that $E^G$ is an essential extension of $S$. Let $e \in E^G$ with $e \neq 0$, so that, by 3.6, $e = (\delta, \sigma_2^{(1)}(\delta), \cdots, \sigma_t^{(1)}(\delta))$ for some $\delta \in K'[(X_1 - a_1)^{-1}, \cdots, (X_n - a_n)^{-1}]$ with $\delta \neq 0$. Let

$$\delta = \sum_{i=1}^{w} k_i'(X_1 - a_1)^{-\alpha_{i1}} \cdots (X_n - a_n)^{-\alpha_{in}},$$

where $w \in \mathbb{N}, k_1', \cdots, k_w' \in K' \backslash \{0\}$, and $(\alpha_{i1}, \cdots, \alpha_{in})(i = 1, \cdots, w)$ are $w$ distinct elements of $\mathbb{N}_0^n$. For each $t = 1, \cdots, n$ , let $m_t$ be the minimal polynomial of $a_t$ over $K$. We can assume that the $n$-tuples $(\alpha_{i1}, \cdots, \alpha_{in})(i = 1, \cdots, w)$ have been ordered so that, for each $i = 1, \cdots, w - 1$, there exits $h_i \in \mathbb{N}$ with $1 \leq h_i \leq n$ such that $\alpha_{ij} = \alpha_{wj}$ for all $j = 1, \cdots, h_i - 1$ and $\alpha_{ih_i} < \alpha_{wh_i}$. Since $m_1(X_1)^{\alpha_{w1}} \cdots m_n(X_n)^{\alpha_{wn}} \in A$, it now follows from 3.6 and ideas like those in the preceding paragraph of this proof that

$$e' = m_1(X_1)^{\alpha_{w1}} \cdots m_n(X_n)^{\alpha_{wn}} e = (k', \sigma_2(k'), \cdots \sigma_t(k'))$$

for some $k' \in K' \backslash \{0\}$. But $K'$ is a finite extension of $K$, and so there exists $f \in A$ such that $k'^{-1} = f(a_1, \cdots, a_n)$. Also, the element $k' \in K'[(X_1 - a_1)^{-1}, \cdots, (X_n - a_n)^{-1}]$ is annihilated by $f - f(a_1, \cdots, a_n)$. In fact, if $f(x_1, \cdots, x_n) = \sum_{i=1}^{v} b_i(X_1 - c_1)^{-\alpha_{i1}} \cdots (X_n - c_n)^{-\alpha_{in}}$, then $fk' = \sum_{i=1}^{v} b_i(X_1 - a_1 + a_1 - c_1)^{-\alpha_{i1}} \cdots (X_n - a_n + a_n - c_n)^{-\alpha_{in}} \cdot k' = f(a_1, \cdots, a_n)k' = 1$. It follows from this that

$$fe' = (k'^{-1}k', \sigma_2(k'^{-1}k'), \cdots, \sigma_t(k'^{-1}k')) = \zeta.$$

Hence $Ae \cap S \neq 0$, and so the proof is complete.

REMARK 3.8. We point out that, in the case in which $K$ has characteristic 0, given a maximal ideal $\mathfrak{m}$ of $K[X_1, \cdots, X_n]$, we can , by 3.3 and 3.1, find a finite, normal field extension $L$ of $K$, with $K \subseteq L \subseteq \bar{K}$, such that $\mathfrak{m}$ splits in $L$ ; we can then use 3.7 to find a description for the injective envelope of the simple $A$-module $A/\mathfrak{m}$.

EXAMPLE 3.9. Let $\mathfrak{m}$ be the maximal ideal $(X^2 - 2, Y^2 + 1)$ in the polynomial ring $\mathbb{Q}[X, Y]$. Let $L = \mathbb{Q}(\sqrt{2}, i)$ and let $E$ be the $L[X, Y]$-module

$$L[(X - \sqrt{2})^{-1}, (Y - i)^{-1}] \oplus L[(X - \sqrt{2})^{-1}, (Y + i)^{-1}] \\ \oplus L[(X + \sqrt{2})^{-1}, (Y - i)^{-1}] \oplus L[(X + \sqrt{2})^{-1}, (Y + i)^{-1}].$$

Then $E_{\mathbb{Q}[X,Y]}(\mathbb{Q}[X, Y]/\mathfrak{m})$ is isomorphic to the $\mathbb{Q}[X, Y]$-submodule of $E$ consisting of all elements which can be written in the form

$$\Bigg(\sum_{j=1}^{w}(a_j + b_j\sqrt{2} + c_j i + d_j i\sqrt{2})(X - \sqrt{2})^{-\alpha_j}(Y - i)^{-\beta_j}, \\ \sum_{j=1}^{w}(a_j + b_j\sqrt{2} - c_j i - d_j i\sqrt{2})(X - \sqrt{2})^{-\alpha_j}(Y + i)^{-\beta_j}, \\ \sum_{j=1}^{w}(a_j - b_j\sqrt{2} + c_j i - d_j i\sqrt{2})(X + \sqrt{2})^{-\alpha_j}(Y - i)^{-\beta_j}, \\ \sum_{j=1}^{w}(a_j - b_j\sqrt{2} - c_j i + d_j i\sqrt{2})(X + \sqrt{2})^{-\alpha_j}(Y + i)^{-\beta_j}\Bigg),$$

where $w$ is a non-negative integer, $(a_j, b_j, c_j, d_j)$ $(j = 1, \cdots, w)$ are $w$ elements of $\mathbb{Q}^4 \backslash \{(0, 0, 0, 0)\}$ and $(\alpha_j, \beta_j)$ $(j = 1, \cdots, w)$ are $w$ distinct elements of $\mathbb{N}_0^2$.

## 4. Connections between ramification indices, residue class degrees and injective modules

In this section, we shall be concerned with ramification indices, residue class degrees and injective envelope of the simple $A$-module $A/\mathfrak{m}$.

**NOTATION 4.1.** Throughout this section, we shall use the following notation. $R$ and $R'$ will denote commutative rings with identity, and $\phi : R \to R'$ will be a ring homomorphism with the property that $R'$, when regarded as an $R$-module by means of $\phi$, is a finitely generated projective $R$-module. We shall use $\mathfrak{p}$ to denote a typical prime ideal of $R$, and $S$ will denote $\phi(R\backslash\mathfrak{p})$, a multiplicatively closed subset of $R'$. We may form the possibly trivial ring $S^{-1}R'$; we shall use $\phi' : R_\mathfrak{p} \to S^{-1}R'$ to denote the ring homomorphism for which $\phi'(r/t) = \phi(r)/\phi(t)$(for $r \in R$ and $t \in R-\mathfrak{p}$). We shall use $k_R(\mathfrak{p})$ to denote the residue field of the local ring $R_\mathfrak{p}$, and $F_{R'}(\mathfrak{p})$ to denote the set of those prime ideals $\mathfrak{q}$ of $R'$ for which $\mathfrak{q} \cap R = \mathfrak{p}$. Finally, if $K_1 \to K_2$ is a homomorphism of fields, we shall use $[K_2 : K_1]$ to denote the (possibly infinite) degree of the extension.

LEMMA 4.2. *([S1,(2,1) and (2,2)])Suppose $\mathfrak{q}$ belongs to $F_{R'}(\mathfrak{p})$, then $\mathfrak{q} \cap S = \emptyset$, and $S^{-1}\mathfrak{q}$ is a prime ideal of $S^{-1}R'$ which contracts under $\phi'$ to $\mathfrak{p}R_\mathfrak{p}$. Furthermore, if $S^{-1}R'$ is not trivial, then $\phi' : R_\mathfrak{p} \to S^{-1}R'$ is injective, and*

$$F_{S^{-1}R'}(\mathfrak{p}R_\mathfrak{p}) = \{S^{-1}\mathfrak{q} : \mathfrak{q} \in F_{R'}(\mathfrak{p})\}$$

*is the set of all maximal ideals of $S^{-1}R'$.*

REMARK AND DEFINITION 4.3. Suppose $\mathfrak{q} \in F_{R'}(\mathfrak{p})$; then $\phi$ induces a natural field homomorphism $\phi_1 : k_R(\mathfrak{p}) \to k_{R'}(\mathfrak{q})$. Also, by 4.2, $S^{-1}\mathfrak{q} \in F_{S^{-1}R'}(\mathfrak{p}R_\mathfrak{p})$; and hence $S^{-1}\mathfrak{q}$ is a maximal ideal of $S^{-1}R'$, and $\phi'$ induces a field homomorphism $\phi'' : k_R(\mathfrak{p}) \to S^{-1}R'/S^{-1}\mathfrak{q}$. Clearly, the degree of this extension is finite.

Next, one can use [Proposition 9, p.153] and [Proposition 19, p.165] of [N1] to construct an isomorphism of fields $\psi : k_{R'}(\mathfrak{q}) \to S^{-1}R'/S^{-1}\mathfrak{q}$ such that $\psi \circ \phi_1 = \phi''$. It follows that $\phi_1$ makes $k_{R'}(\mathfrak{q})$ into a finite extension of $k_R(\mathfrak{p})$ ; we shall denote the degree of this extension by $f(\mathfrak{q}/\mathfrak{p})$ and refer to this as the *residue class degree of* $\mathfrak{q}$ *over* $\mathfrak{p}$.

It follows that

$$f(\mathfrak{q}/\mathfrak{p}) = [S^{-1}R'/S^{-1}\mathfrak{q} : k_R(\mathfrak{p})].$$

LEMMA 4.4. *([CE, II, Proposition 6. 1a.]) Let $E$ be an injective $R$-module. Then $Hom_R(R', E)$, when regarded as a $R'$-module in the natural way, is $R'$-injective.*

PROPOSITION 4.5. *([S1, (2.5)]) Suppose that $R$ is local with maximal ideal $\mathfrak{p}$. Then*

(i) *$F_{R'}(\mathfrak{p})$ is a finite set;*
(ii) *for each $\mathfrak{q} \in F_{R'}(\mathfrak{p})$, the $R'_{\mathfrak{q}}$ -module $R'_{\mathfrak{q}}/\mathfrak{p}R'_{\mathfrak{q}}$ has finite length.*

COROLLARY AND DEFINITION 4.6. *Suppose $\mathfrak{q}$ is a prime ideal in $F_{R'}(\mathfrak{p})$. Then $R'_{\mathfrak{q}}/\mathfrak{p}R'_{\mathfrak{q}}$ is a non-zero $R'_{\mathfrak{q}}$-module of finite length: we denote this length by $e(\mathfrak{q}/\mathfrak{p})$, and call this the generalized ramification index of $\mathfrak{q}$ over $\mathfrak{p}$.*

For the following proposition, we shall assume in addition that the rings $R$ and $R'$ are Noetherian. Let $\mathfrak{p}$ be a prime ideal of $R$ and $\mathfrak{q} \in F_{R'}(\mathfrak{p})$. Note that (since $R$ and $R'$ are Noetherian) we can see easily (without recourse to 4.5) that $F_{R'}(\mathfrak{p})$ is a finite set and, for each $\mathfrak{q} \in F_{R'}(\mathfrak{p})$, the $R'_{\mathfrak{q}}$-module $R'_{\mathfrak{q}}/\mathfrak{p}R'_{\mathfrak{q}}$ has finite length.

PROPOSITION 4.7. *[S1, (4.2) and (4.3)]*

(i) $Hom_R(R', E_R(R/\mathfrak{p})) \cong \bigoplus_{\mathfrak{q} \in F_{R'}(\mathfrak{p})} E_{R'}(R'/\mathfrak{q})$ *(as $R'$-modules);*
(ii) *Let $\mathfrak{q}$ be a prime in $F_{R'}(\mathfrak{p})$. Then*

$$E_{R'}(R'/\mathfrak{q}) \cong \bigoplus e(\mathfrak{q}/\mathfrak{p})f(\mathfrak{q}/\mathfrak{p})E_R(R/\mathfrak{p})(\text{as } R\text{-modules}).$$

THEOREM 4.8. *Let the situation and notation be as in 3.1, 3.3 and 3.7. Then*

$$\begin{aligned} &Hom_{K[X_1,\cdots,X_n]}(L[X_1,\cdots,X_n], E^G) \\ &\cong \bigoplus_{\mathfrak{m}_i \in \{\mathfrak{m}_1,\cdots,\mathfrak{m}_t\}} \{\oplus e(\mathfrak{m}_i/\mathfrak{m})f(\mathfrak{m}_i/\mathfrak{m})E^G\} \\ &\qquad (\text{as } K[X_1,\cdots,X_n]\text{-modules}), \end{aligned}$$

*where $e(\mathfrak{m}_i/\mathfrak{m})$ is the generalized ramification index of $\mathfrak{m}_i$ over $\mathfrak{m}$ and, $f(\mathfrak{m}_i/\mathfrak{m})$ is the residue class degree of $\mathfrak{m}_i$ over $\mathfrak{m}$.*

*Proof.* It is clear that the natural $K$-algebra homomorphism $\phi : A \to B = L \otimes_K A$ makes $B$ (faithfully) flat over $A$. Thus, for each $\mathfrak{m}_i \in \{\mathfrak{m}_1, \cdots, \mathfrak{m}_t\}$, $E_B(B/\mathfrak{m}_i)$, when regarded as a $A$-module by means of $\phi$, is isomorphic to a direct sum of copies of $E_A(A/\mathfrak{m})$. By 4.7(ii), we have

$$(1) \qquad E_B(B/\mathfrak{m}_i) \cong \bigoplus e(\mathfrak{m}_i/\mathfrak{m}) f(\mathfrak{m}_i/\mathfrak{m}) E^G \text{ (as } A\text{-modules)},$$

where $e(\mathfrak{m}_i/\mathfrak{m})$ is the generalized ramification index of $\mathfrak{m}_i$ over $\mathfrak{m}$ and, $f(\mathfrak{m}_i/\mathfrak{m})$ is the residue class degree of $\mathfrak{m}_i$ over $\mathfrak{m}$.

Now, it follows from 4.4 that $\mathrm{Hom}_A(B, E^G)$, when regarded as a $B$-module in the natural way, is $B$-injective. So, by 4.7(i), there is an $B$-isomorphism

$$(2) \qquad \mathrm{Hom}_A(B, E^G) \cong E.$$

¿From (1) and (2), we get

$$\mathrm{Hom}_A(B, E^G) \cong \bigoplus_{\mathfrak{m}_i \in \{\mathfrak{m}_1, \cdots, \mathfrak{m}_t\}} \{\oplus e(\mathfrak{m}_i/\mathfrak{m}) f(\mathfrak{m}_i/\mathfrak{m}) E^G\},$$

as required.

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