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TOTALLY UMBILIC HYPERSURFACES OF A SPACE FORM

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1. Introduction

A totally umbilic submanifold of a pseudo-Riemannian manifold is a submanifold whose first fundamental form and second fundamental form are propotional. An ordinary hypersphere $S^n(r)$ of an affine $(n+1)$-space of the Euclidean space E^m is the best known example of totally umbilic submanifolds of E^m. The totally umbilic submanifolds of a Riemannian space form with constant sectional curvature are well known ([3,4]), For totally umbilic submanifolds of pseudo-Riemannian space form, see [1] and [8].

An infinitesimal conformal transformation, or conformal vector field on a pseudo-Riemannian manifold (M, g) is a vector field V on M satisfying $\mathfrak{L}_V g = 2\sigma g$, where $\mathfrak{L}$ denotes the Lie derivative on M and σ is a smooth function. If (M, g) is a totally umbilic submanifold of a pseudo-Riemannian manifold $(\bar{M}, \bar{g})$, then it is well-known that for any conformal vector field V on $\bar{M}$, the tangential part V^T of V on M becomes a conformal vector field on M(Proposition in § 2).

In this note we prove the converse of the above proposition for a hypersurface of a pseudo-Riemannian space form $\bar{M}^{n+1}_\nu(\bar{c})$ with constant sectional curvature $\bar{c}$.

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2. Main Theorem

On a pseudo-Riemannian manifold (M, g) a vector field V is called conformal if it preserves the conformal class of the metric :

$$\mathfrak{L}_V g = 2\sigma g$$

for some function σ.
Recall that by definition $\mathfrak{L}_V g(X,Y) = g(\nabla_X V, Y) + g(X, \nabla_Y V)$ for arbitrary tangent vectors X, Y where ∇ denotes the Levi-civita connection. Necessarily the function σ is $\frac{1}{n}div(V)$, where $div(V)$ denotes the divergence of the vector field V.

PROPOSITION. *Let (M^n, g) be a totally umbilic submanifold of a pseudo-Riemannian space $(\bar{M}^m, \bar{g})$. If V is a conformal vector field on $\bar{M}$, then the tangential part V^T of V on M is a conformal vector field on M.*

proof. Let V^T and V^N be the tangential and normal part of V on M, respectively. Then since for all $X, Y \in TM$

$$\begin{aligned}\mathfrak{L}_V \bar{g}(X,Y) &= \bar{g}(\bar{\nabla}_X V, Y) + \bar{g}(X, \bar{\nabla}_Y V)\\ &= \bar{g}(\bar{\nabla}_X V^T, Y) + \bar{g}(X, \bar{\nabla}_Y V^T) + \bar{g}(\bar{\nabla}_X V^N, Y) + \bar{g}(X, \bar{\nabla}_Y V^N),\end{aligned}$$

we obtain from the hypothesis that

$$\begin{aligned}\mathfrak{L}_V \bar{g}(X,Y) &= g(\nabla_X V^T, Y) + g(X, \nabla_Y V^T) - 2\bar{g}(V^N, H) g(X,Y)\\ &= \mathfrak{L}_{V^T} g(X,Y) - 2\bar{g}(V,H) g(X,Y).\end{aligned}$$

Hence we see that for all $X, Y \in TM$

$$\begin{aligned}\mathfrak{L}_{V^T} g(X,Y) &= \mathfrak{L}_V \bar{g}(X,Y) + 2\bar{g}(V,H) g(X,Y).\\ &= 2\{\sigma + \bar{g}(V,H)\} g(X,Y),\end{aligned} \tag{2.1}$$

where $n\sigma$ is the divergence of V on $\bar{M}$. This completes the proof.

Now for a hypersurface of a pseudo-Riemannian space form we prove the converse as follows :

THEOREM. *Let (M^n, g) be a connected hypersurface of a pseudo-Riemannian space form $(\bar{M}_\nu^{n+1}(\bar{c}), \bar{g})$. Suppose that $\bar{M}$ carries a conformal vector field V with $\mathfrak{L}_V \bar{g} = 2\sigma\bar{g}$ of which the tangential part V^T on M becomes a conformal vector field with $\mathfrak{L}_{V^T} g = 2\tau g$. If the restriction $\sigma|_M$ of σ on M is not identically equal to τ, then (M^n, g) is a totally umbilic and not totally geodesic submanifold of $(\bar{M}_\nu^{n+1}(\bar{c}), \bar{g})$.*

proof. As in the proof of the above proposition, we obtain

$$\mathfrak{L}_{V^T} g(X,Y) = \mathfrak{L}_V \bar{g}(X,Y) + 2\bar{g}(V, h(X,Y)) \tag{2.2}$$

for all $X, Y \in TM$, where h is the second fundamental form of M in $\bar{M}$. From the hypothesis and from (2.2) we see that for all $X, Y \in TM$

$$\bar{g}(V, h(X,Y)) = (\tau - \sigma) g(X,Y). \tag{2.3}$$

We let $U = \{p \in M | \sigma(p) \neq \tau(p)\}$, then U is a nonempty open set. And (2.3) shows that U is totally umbilic with mean curvature vector field $H = \frac{\tau-\sigma}{<V,\xi>}\xi$, where ξ a locally defined unit normal vector field on M with $\bar{g}(\xi,\xi) = \epsilon = \pm 1$. Hence by Codazzi equation we see that for each connected component U_i of U, there exists a nonzero constant a_i which satisfies $\tau - \sigma = a_i < V, \xi >$ and $H = a_i\xi$ on U_i. Furthermore, each U_i has constant sectional curvature $c_i = \bar{c} + \epsilon a_i^2$. Since V is a conformal vector field on a space form $\bar{M}_\nu^{n+1}(\bar{c})$, the divergence $(n+1)\sigma$ satisfies ([6,9])

$$\bar{\nabla}_X \bar{\nabla}\sigma = -\bar{c}\sigma X \tag{2.4}$$

for all vector field X on $\bar{M}$, where $\bar{\nabla}\sigma$ denotes the gradient of σ on $\bar{M}$. Analogously, on each U_i, τ satisfies

$$\nabla_X \nabla\tau = -c_i \tau X \tag{2.5}$$

for all vector field X on U_i.

From (2.4), it is easy to show that on each U_i

$$\nabla_X \nabla\sigma = -\varphi X \tag{2.6}$$

for all vector field X on U_i, where we denote by φ and $\nabla\sigma$ the function $\bar{c}\sigma - \epsilon a_i < \bar{\nabla}\sigma, \xi >$ and the gradient of the restriction $\sigma|_M$ on U_i, respectively. From (2.6) it follows that for all vector fields X, Y on U_i

$$(2.7) \qquad R(X,Y)\nabla\sigma = < Y, \nabla\varphi > X - < X, \nabla\varphi > Y,$$

where R is the Riemann curvature tensor of (M^n, g). Since U_i has constant sectional curvature c_i, (2.7) shows that $\nabla\varphi = c_i\nabla\sigma$, that is, $\varphi = c_i\sigma + b_i$ for some constant b_i. Hence (2.6) may be rewritten as follows :

$$(2.8) \qquad \nabla_X\nabla\sigma = -(c_i\sigma + b_i)X$$

for all vector fields X on U_i. Thus from (2.5) and (2.8) we have for all tangent vector fields X on U_i

$$(2.9) \qquad \nabla_X\nabla(\tau - \sigma) = -\{c_i(\tau - \sigma) - b_i\}X.$$

Now suppose that the interior open set W of U^c is not empty. Then on W, hence on the closure $\bar{W}$ of W we have

$$(2.10) \qquad \tau - \sigma = 0, \quad \nabla(\tau - \sigma) = 0, \quad \nabla_X\nabla(\tau - \sigma) = 0.$$

Since $\bar{W}$ is a proper subset of M, there exists at least one component U_i of U of which closure intersects $\bar{W}$. For such U_i, (2.9) and (2.10) show that $b_i = 0$. By the same argument as above, it may be proven that b_i is trivial for all i. If $\bar{U}_i$ intersects $\bar{U}_j$, then on $\bar{U}_i \cup \bar{U}_j$ M has mean curvature vector field $H = a_i\xi = a_j\xi$, and has constant sectional curvature, $c_i = c_j$. This implies that if we let A_i denote a connected component of the complement of $\bar{W}$, then we have for all vectors X on A_i

$$(2.11) \qquad \nabla_X\nabla(\tau - \sigma) = -c_i(\tau - \sigma)X,$$

where c_i denotes the constant sectional curvature on A_i. For a fixed point p in the boundary of W, we consider a normal neighborhood N about p. Then on N, there exists a point q of U. Since U is contained in the complement of $\bar{W}$, q lies in a component A_i. Let $\gamma(t)$ denote the unique geodesic in N with $\gamma(0) = p$ and $\gamma(1) = q$, and let t_o denote the infimum of t which satisfies $\gamma([t, 1]) \subset A_i$. Then, since $\gamma(t_o)$ lies in the boundary of W, $(\tau - \sigma)$ and $\nabla(\tau - \sigma)$ vanish at $\gamma(t_o)$, respectively. Hence (2.11) with Proposition 2.1 in [7] shows that $\tau - \sigma$ vanishes identically on $\gamma([t_o, 1])$, in particular, $(\tau - \sigma)$ vanishes at q in U. This contradiction shows that the interior open set W of U^c is empty, that is, the closure $\bar{U}$ of U is the whole hypersurface M. This, by continuity, completes the proof.

REMARK. *If M^n is a totally geodesic hypersurface or an integral hypersurface of V (that is, $V|_M \in TM$), then (2.1) shows that the tangential part V^T on M is conformal on M with $\sigma|_M = \tau$.*

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