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ON THE ADMITTANCE OF A FIXED POINT FREE DEFORMATION OF THE SPACE WHICH $\pi_1(X)$ IS INFINITE

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Abstract In this paper, we shall investigate the admittance of a fixed point free deformation($FPFD$) on the locally nilpotent spaces when $\pi_1(X)$ is infinite. More precisely, for $X \in (S_{*LN})$ with $\pi_1(X)$ infinite, we prove the admittance of a $FPFD$ where $\pi_1(X)$ has the maximal condition on normal subgroups, or $\pi_1(X)$ satisfies either the max-∞ or min-∞ for non-nilpotent subgroups where S_{*LN} denotes the category of the locally nilpotent spaces and base point preserving continuous maps.

1. Introduction

There are many results on the nilpotent spaces and its applications which have been studied by G.Mislin [11], A.K.Bousfield [1], P.Hilton [6], S.E.Han [5], and the others. We know that the locally nilpotent space X admits a fixed point free deformation if X is finite type and $\pi_1(X)$ is finite [9, Theorem 4.3, Theorem 4.5].

Hence it is natural to ask the following; in case $\pi_1(X)$ is infinite under what condition is there admittance of the $FPFD$? In this paper, when $\pi_1(X)$ is infinite, we investigate the admittance of a $FPFD$ of the space X such that $\pi_1(X)$ has the maximal condition on normal subgroups, or $\pi_1(X)$ satisfies either max-∞ or min-∞ for non-nilpotent subgroups.

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We work in the category of the connected CW-complexes with base point and we denote the category by S_*.

2. Preliminaries

In this section, for convenience we repeat here the definitions and basic properties of the locally nilpotent space and some properties with relation to the fixed point theory.

We recall that a locally nilpotent group is the group whose all finitely generated subgroups are nilpotent groups [12].

And we denote the category of nilpotent spaces and continuous maps by S_{*N}.

Now we extend the concept of nilpotent space as follows; a space $X(\in S_*)$ is said to be a locally nilpotent space if
(1) $\pi_1(X)$ is a locally nilpotent group,
(2) $\pi_1(X)$ acts on $\pi_n(X)$ nilpotently for all $n \geq 2$ [13].

And we denote the category of locally nilpotent spaces and continuous maps by S_{*LN}.

For any nilpotent groups, their finitely generated subgroups are also nilpotent groups, thus the category S_{*N} is the full subcategory of S_{*LN}[8].

LEMMA 2.1. *[10] A space X is nilpotent if and only if $\pi_1(X)$ is a nilpotent group and $\pi_1(X)$-module $H_i(\tilde{X};\mathbb{Z})$ are nilpotent for all $i \geq 0$, where $\tilde{X}$ is the universal covering space of X.*

DEFINITION 2.2. [12] A group G satisfies the maximal condition if it has no infinite strictly increasing chain of subgroups.

LEMMA 2.3. *[2] If X is a polyhedron and Euler characteristic $\chi(X) = 0$, then X admits a $FPFD$.*

LEMMA 2.4. *[3] For finite X, if $\pi_1(X)$ contains a torsion free nontrivial normal abelian subgroup which acts nilpotently on $H_*(\tilde{X})$, then $\chi(X) = 0$.*

3. Main Results

We are now prepared to establish theorems solving the problem already posed in the Introduction. We shall study the admittance of a $FPFD$ of the locally nilpotent space such that $\pi_1(X)$ has the maximal condition on the normal subgroup, or $\pi_1(X)$ satisfies either max-∞ or min-∞ for non-nilpotent subgroups when $\pi_1(X)$ is infinite.

Given a property ★ pertaining to subgroups, a group G is said to satisfy max-∞ for ★-subgroups if G has no infinite ascending chain $H_1 \subset H_2 \subset \cdots$ of ★-subgroups in which all indices $|H_{i+1} : H_i|$ are infinite. The property min-∞ for ★ -subgroups is defined similarly [7].

Let G be the torsion free locally nilpotent group and H a subgroup of G. For each set π of primes, the π-isolator of H in G, which is the set $\{g \in G : g^n \in H \text{ for some } n \in \pi\}$, is a subgroup of G. In the case where π is the set of all primes we refer simply to the isolator of H in G, denoted $I_G(H)$, and H is said to be isolated in G if $I_G(H) = H$. If H is countable then so is $I_G(H)$; this is an easy consequene of the fact that, for $x, y \in G$ and $n \in \mathbb{N}, x^n = y^n$ implies $x = y$. If H is nilpotent of class c then so is $I_G(H)$. Finally, if G is finitely genterated and $\{N_i : i = 1, 2, \cdots\}$ is the set of all normal subgroups of finite index in G then $H = \cap_{i=1}^{\infty} HN_i$

THEOREM 3.1. *For $X \in S_{*LN}$, with $\pi_1(X)$ infinite if either (1) $\pi_1(X)$ has the maximal condition on normal subgroups or*
(2) $\pi_1(X)$ is a finitely generated group which satisfies either max-∞ or min-∞ for non-nilpotent subgroups.
Then X admits a $FPFD$.

Proof. (Case 1): when $\pi_1(X)$ satisfies the maximal condition on normal subgroups, $\pi_1(X)$ is a finitely generated nilpotent group. Hence $X \in S_{*N}$.

(Case 2): when $\pi_1(X)$ satisfies max-∞ or min-∞, for non-nilpotent subgroups, assume that $\pi_1(X)$ is not nilpotent. Then $\pi_1(X)$ has a countable non-nilpotent subgroups and hence such an isolated subgroup, which we denote by K. Write $K = \cup\{K_i\}_{i=1}^{\infty}$, where $1 = K_0 \subset K_1 \subset K_2 \subset \cdots$ is a chain of finitely generated subgroups of increasing nilpotency classes. Let $\{p_1, p_2, p_3, \cdots\}$ be an infinite set of primes. Choose a normal subgroup H_1 of K_1 such that the index $|K_1 : H_1|$ is finite and divisible by p_1.

Now let N_2 be a normal subgroup of finite index in K_2 such that $|K_2 : N_2K_1|$ is divisible by p_2 and $N_2H_1 \cap K_1 = H_1$. Write $H_2 = N_2H_1$. Inductively, having defined N_i and H_i for some $i \geq 2$, let N_{i+1} be a normal subgroup of finite index in K_{i+1} such that $|K_{i+1} : N_{i+1}K_i|$ is divisible by p_{i+1} and $N_{i+1}H_i \cap K_i = H_i$ and write $H_{i+1} = N_{i+1}H_i$. We obtain an infinite chain $H_1 \subset H_2 \subset \cdots$ such that, in particular, $|K_i : H_i|$ is finite for each i. Thus, setting $H = \cup_{i=1}^{\infty} H_i$. We have $I_K(H) = K$ and hence H non-nilpotent. Now we define $L_0 = H$ and for each $i \geq 1, L_i =< H, K_i >$. We shall establish the following facts:

(1) For each $i \geq 1, H \cap K_i = H_i$

(2) $|L_1 : L_0| = |K_1 : H_1|$ and, for $i \geq 1$ $|L_{i+1} : L_i| = |K_{i+1} : N_{i+1}K_i|$.

From the choice of the subgroups N_i, we see that $|L_{i+1} : L_i|$ is divisible by p_{i+1} for all $i \geq 1$. In particular, the chain $L_0 \subset L_1 \subset \cdots$ is not finite. We now obtain a similar chain where the indices are all infinite. Let π denote the set of all primes p which divide at least one of the indices $|L_{i+1} : L_i|$. Then π is infinite and we may write it as a disjoint union of infinitely many infinite subsets $\{\pi_i\}_{i=1}^{\infty}$. Let I_1 denote the π_1 -isolator of H in K and, for $n \geq 1$, let I_{n+1} denote the π_{n+1} -isolator of I_n in K. Then each of the indices $|I_{n+1} : I_n|$ is infinite. Thus $\pi_1(X)$ does not satisfy max-∞ for non-nilpotent subgroups. In the other hand, if J_1 denotes the π_1' -isolator of H in K, ($'$ means the complement) for $n \geq 1, J_{n+1}$ denotes the π_{n+1}'-isolator of H in J_n, then each of the indices $|J_n : J_{n+1}|$ is infinite and $\pi_1(X)$ does not satisfy min-∞ for non-nilpotent subgroups. We thus have a contradiction.

Anyway, even though we follow the any case (1) or (2), by Lemma 2.1, $\pi_1(X)$ acts nilpotently on $H_i(\tilde{X})$. Hence the center $Z(\pi_1(X))$ of $\pi_1(X)$ is infinite and finitely generated. Then we can take an infinite cyclic subgroup $Z(\pi_1(X))$ for a torsion free nontrivial normal abelian subgroup of Lemma 2.4. Thus $\chi(X) = 0$. Hence X admits a $FPFD$.

THEOREM 3.2. *For finite $X \in S_{*LN}, Y(\in S_*)$ and if $\pi_1(X)$ is infinite with the maximal condition on normal subgroups and $\pi_1(X)$ satisfies either max-∞ or min-∞ for non-nilpotent subgroups, then Y also admits a $FPFD$, if $f : X \to Y$ is an acyclic* *map.*

Proof. (Case 1): since $\pi_1(X)$ is infinite and has the maximal condition on normal subgroups, $\pi_1(X)$ is a finitely generated nilpotent group. Hence $X \in S_{*N}$.

(Case2): if $\pi_1(X)$ satisfies either max-∞ or min-∞, for non-nilpotent subgroups, by the (Case 2) of the Theorem 3.1, we get $X \in S_N$

From the fact that $f : X \to Y$ is an acyclic map and the classical homotopy exact sequence of fibration: $F_f \to X \to Y$, we know that $\pi_1(f)$ is an epimorphism. Furthermore $H_1(F_f) \cong \frac{\pi_1(F_f)}{[\pi_1(F_f),\pi_1(F_f)]} = 0$ where $[,]$ means the commutator subgroup and F_f is the homotopy fiber of f. Thus $P\pi_1(X)$ is perfect normal subgroup of $\pi_1(X)$. Since $X \in S_{*N}$, $P\pi_1(X)$ is trivial. Thus $\pi_1(f) : \pi_1(X) \to \pi_1(Y)$ is an isomorphism. By use of the Hurewicz Theorem [4] inductively, $\pi_i(F_f) = 0$. Thus f is a weak homotopy equivalence. By the Whitehead Theorem [4], f is a homotopy equivalence. Since the Euler characteristic number is invariant under the homotopy equivalence, thus by Theorem 3.1, our proof is completed.

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