쉬프트를 갖는 부분공간 반복법의 개선

Improvement of Subspace Iteration Method with Shift

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요 약 : 본 논문에서는 쉬프트를 갖는 부분공간 반복법의 제한조건을 제거하여 수치적으로 안정한 고유치해석 방법을 제안 하였다. 쉬프트를 갖는 부분공간 반복법의 주된 단점은 특이성 문제 때문에 어떤 고유치에 근접한 쉬프트를 사용할 수 없어서 수렴성이 저하될 가능성이 있다는 점이다. 본 논문에서는 부가조건식을 이용하여 위와 같은 특이성 문제를 수렴성의 저하없이 해결하였다. 이 방법은 쉬프트가 어떤 고유치와 같은 경우일지라도 항상 비특이성인 성질을 갖고 있다. 이것은 제안방법의 중요한 특성중의 하나이다. 제안방법의 비특이성은 해석적으로 증명되었다. 제안방법의 수렴성은 쉬프트를 갖는 부분공간 반복법의 수렴성과 거의 같고, 두 방법의 연산횟수는 구하고자 하는 고유치의 개수가 많은 경우에 거의 같다. 제안방법의 효율성을 증명하기 위하여, 두개의 수치예제를 고려하였다.

ABSTRACT : A numerically stable technique to remove the limitation in choosing a shift in the subspace iteration method with shift is presented. A major difficulty of the subspace iteration method with shift is that because of singularity problem, a shift close to an eigenvalue can not be used, resulting in slower convergence. This study solves the above singularity problem using side conditions without sacrifice of convergence. The method is always nonsingular even if a shift is an eigenvalue itself. This is one of the significant characteristics of the proposed method. The nonsingularity is proved analytically. The convergence of the proposed method is at least equal to that of the subspace iteration method with shift, and the operation counts of above two methods are almost the same when a large number of eigenpairs are required. To show the effectiveness of the proposed method, two numerical examples are considered.

핵 심 용 어 : 부분공간 반복법, 쉬프트, 비특이성, 부가조건식

KEYWORDS : Subspace iteration method, Shift, Nonsingularity, Side condition

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1. Introduction

Eigenvalue analysis is an important step in structural dynamic analysis when the mode superposition method is used. Many solution methods have been developed for eigenvalue analysis, and among these methods the subspace iteration method has hitherto been known to be very efficient for solving large eigenvalue problems.

The subspace iteration method was developed and named by Bathe[1-2]. This method combines simultaneous inverse iteration method and Rayleigh-Ritz analysis. This method has been widely used, but the following shortcomings have been identified after extensive use of the method[3].

(1) When the number of eigenpairs to be required is large, the convergence of the required eigenvalues can be very slow.

(2) If a large number of eigenpairs are required, the computational effort required to form and solve the subspace eigenvalue problem can be significant.

(3) When the starting iteration vectors are poorly chosen, some of the eigenvalues and corresponding eigenvectors of interest may be missed.

To overcome the above shortcomings many researchers have studied a variety of acceleration procedures of the subspace iteration method as follows.

Yamamoto and Ohtsubo[4] have used Chebyshev polynomials for acceleration and they have shown that improved convergence in the subspace iteration can result. Akl et al.[5] have employed over-relaxation method to accelerate the subspace iteration and they have demonstrated the effectiveness of the method. Bathe and Ramaswamy[6] have used over-relaxation and shifting techniques and they showed that the accelerated method can be applied effectively to the solution of eigenproblems in which the matrices have small or large bandwidths. Nguyen and Arora[7] have developed the method for free vibration analysis of a large structure by partitioning it into a number of substructures to reduce the computer storage requirement. Cheu et al.[8] have investigated the effects of selecting initial vectors on computation efficiency for subspace iteration method. Lam and Bertolini[9] have developed selective repeated inverse iteration and multiple inverse iteration for accelerated reduction of subspace.

Rajendran and Narasimhan[10] have used another over-relaxation method. The method proposed by Bathe and Ramaswamy (1980) considers the acceleration of individual vectors using individual over-relaxation parameters, whereas the method proposed by Rajendran and Narasimhan considers the acceleration of the subspace as a whole. Qian and Dhatt[11] have accelerated the subspace iteration by omitting some of the Rayleigh-Ritz procedure from certain iteration steps and obtaining a higher convergence rate.

Among the above accelerated techniques, a shifting technique is effectively used in

the commercial FEM programs such as ADINA[12]. Since the singularity may occur during the use of the shifting technique in the accelerated scheme such as the subspace iteration method with shift, the shift must be within a limited region to avoid the singularity.

This paper describes a technique which always guarantees the numerical stability and maintains the convergence rate of the subspace iteration method with shift even if it is an exact eigenvalue itself. The theory and concept of the proposed method are discussed briefly, and two numerical examples are presented to verify the effectiveness of the proposed method.

2. Subspace Iteration Method with Shift

The general eigenvalue problem of the structural dynamics may be written as follows.[13]

$$KX = MX\Lambda \tag{1}$$

where K and M are the stiffness matrix and the mass matrix of the discrete or discretized system of order n, respectively, the columns of X the eigenvectors, and Λ a diagonal matrix with eigenvalues.

Applying a shift μ to eqn (1) gives

$$(K - \mu M)X = MX\Omega \tag{2}$$

where

$$\Omega = \Lambda - \mu I \tag{3}$$

and I is the unit matrix. Suppose that the p smallest eigenvalues λ_i $(i=1,2,\ldots,p)$ and corresponding eigenvectors x_i are required. Then the jth trial vector converges linearly to x_j at the rate of $(\lambda_j - \mu)/(\lambda_{p+1} - \mu)$. For faster convergence, q trial vectors are normally used with $q = \min\{2p, p+8\}$.

If we have p initial independent vectors $x_i^{(0)}$ $(i=1,2,\ldots,p)$ spanning p-dimensional subspace in the neighborhood of the subspace of the desired eigenvectors and the approximate eigenvectors and corresponding eigenvalues after k iterations are denoted by $x_i^{(k)}$ and $\lambda_i^{(k)}$, the subspace iteration method with the shift μ for the kth iteration may be described as follows:

Step 1. Find improved eigenvectors $\overline{X}^{(k+1)} = [\ \overline{x}_1^{(k+1)},\ \overline{x}_2^{(k+1)}, \ldots,\ \overline{x}_q^{(k+1)}]$ by the simultaneous inverse iteration method:

$$(K - \mu M)\ \overline{X}^{(k+1)} = MX^{(k)} \tag{4}$$

where $\overline{X}^{(k+1)}$ and $X^{(k)}$ are the $(n \times q)$ matrices.

Step 2. Compute the projections of the matrices $(K - \mu M)$ and M onto the subspace spanned by the q vectors in $\overline{X}^{(k+1)}$:

$$\overline{K}^{(k+1)} = \overline{X}^{(k+1)^T}(K - \mu M)\ \overline{X}^{(k+1)} \tag{5}$$

$$\overline{M}^{(k+1)} = \overline{X}^{(k+1)^T} M\ \overline{X}^{(k+1)} \tag{6}$$

where $\overline{K}^{(k+1)}$ and $\overline{M}^{(k+1)}$ are the symmetric matrices. Step 3. Solve the eigenvalue problem of reduced order q:

$$\overline{K}^{(k+1)} Q^{(k+1)} = \overline{M}^{(k+1)} Q^{(k+1)} \Omega^{(k+1)} \quad (7)$$

where $Q^{(k+1)}$ and $\Omega^{(k+1)}$ are the $(q \times q)$ matrices.

Step 4. Find an improved approximation to the eigenvectors $X^{(k+1)}$ from $\overline{X}^{(k+1)}$, the $(n \times q)$ matrix of Ritz trial vectors, and the $(q \times q)$ projected system eigenvectors $Q^{(k+1)}$:

$$X^{(k+1)} = \overline{X}^{(k+1)} Q^{(k+1)} \quad (8)$$

And the improved eigenvalues can be computed as follows.

$$\Lambda^{(k+1)} = \Omega^{(k+1)} + \mu I \quad (9)$$

Then, provided that the trial vectors in $X^{(1)}$ are not orthogonal to one of the required eigenvectors and assuming an appropriate ordering of the trial vectors, $\Lambda^{(k+1)}$ converges to Λ and $X^{(k+1)}$ converges to X as k approaches infinity. The convergence rate of the subspace iteration method with shift is

$$(\lambda_j - \mu)/(\lambda_{q+1} - \mu). \quad (10)$$

While the shifting procedure improves the convergence rate of the subspace iteration method, it needs extra operations. Therefore, shifting will only be performed when a criterion determines that the convergence will be improved sufficiently to cover the cost of the extra triangular factorization[6].

If a shift is an eigenvalue itself or very close to it, all iteration vectors immediately converge to the eigenvector corresponding to that eigenvalue. The iteration vectors can then not be orthogonalized any more and the iteration procedure becomes unstable. If the shift is very close to an eigenvalue, the last pivot element in the LDL^T factorization of the coefficient matrix usually becomes small compared with its original value and the coefficient matrix becomes close to singular. To avoid this singularity, that is, to guarantee the stability of the subspace iteration method with shift, the following condition was adopted in the subspace iteration method[6]:

$$1.01\ \overline{\lambda}_{s-1} \leq \mu \leq 0.99\ \overline{\lambda}_s \quad (11)$$

where $\overline{\lambda}_{s-1}$ is the calculated approximation to (s-1)th eigenvalue and $\overline{\lambda}_s$ sth eigenvalue. It means that a shift must be within a limited region resulting in slow convergence. Moreover, if the calculated approximation to an eigenvalue slightly differs from it, an eigenvalue may be inside of the limited region. Then, the singularity may occur although a shift is inside of the limited region. These are the significant disadvantages of the subspace

iteration method with shift. The purpose of this paper is to remove the limitation in eqn (11) for choosing the value of a shift μ.

3. Proposed Method

3.1 Theory

Consider the simultaneous inverse iteration step in the subspace iteration method with shift:

$$(K-\mu M)\ \overline{X}^{(k+1)} = MX^{(k)} \tag{12}$$

Since if a shift is very close to an eigenvalue in eqn (12) the singularity occurs during the decomposition process, The (k+1)th eigenvector approximations, $\overline{X}^{(k+1)}$, can not be acquired. In this study, to solve the singularity problem the following procedures are proposed.

First, assume that a shift is close to ith eigenvalue. Then, inverse iteration step on the ith eigenvalue can be expressed as follows:

$$(K-\mu M)\ \overline{x}_i^{(k+1)} = d_{ii}^{(k+1)} M x_i^{(k)} \tag{13}$$

where the scalar $d_{ii}^{(k+1)}$ controls the length of the vector $\overline{x}_i^{(k+1)}$. Because there are only n equations with (n+1) unknowns, n components of $\overline{x}_i^{(k+1)}$ and $d_{ii}^{(k+1)}$, in eqn (13), one side condition must be introduced for the solution of eqn (13). This condition is that the current vector($x_i^{(k)}$)is orthogonal to the incremental vector($\Delta x_i^{(k)}$) with respect to M: that is,

$$x_i^{(k)T} M \Delta x_i^{(k)} = 0. \tag{14}$$

Adding the mass orthonormality relation, $x_i^{(k)T} M\, x_i^{(k)} = 1$, to the side condition, eqn (14), yields

$$x_i^{(k)T} M\, \overline{x}_i^{(k+1)} = 1 \tag{15}$$

where

$$\overline{x}_i^{(k+1)} = x_i^{(k)} + \Delta x_i^{(k)}. \tag{16}$$

The inverse iteration step on the other eigenvalues make use of the existing equation, eqn (12): that is,

$$(K-\mu M)\ \overline{x}_j^{(k+1)} = Mx_j^{(k)} \quad (j=1,2,\ldots,q,\ j\neq i) \tag{17}$$

Writing eqns (13), (15) and (17) in matrix form gives

$$\begin{bmatrix} K-\mu M & Mx_i^{(k)} \\ x_i^{(k)T} M & 0 \end{bmatrix} \begin{bmatrix} \overline{X}^{(k+1)} \\ d_i^{(k+1} \end{bmatrix} = \begin{bmatrix} MX^{(k)} \\ e_i \end{bmatrix} \tag{18}$$

where

$$x_i^{(k)T} M\, \overline{x}_j^{(k+1)} = 0 \ \ (j=1,2,\ldots,q,\ j\neq i)$$

, $d_i^{(k+1)} = \langle 0, \ldots, 0, (1-d_{ii}^{(k+1)}), 0, \ldots, 0 \rangle$ is the row vector of order q, and $e_i = \langle 0, \ldots, 0, 1, 0, \ldots, 0 \rangle$ is the row vector of order q that all elements are zero except for ith element with unity.

Note that $\overline{X}^{(k+1)}$ from eqn (18) is used for $\overline{X}^{(k+1)}$ in eqns (5) and (6) instead of $\overline{X}^{(k+1)}$ in eqn (4). Eqn (18) is the main linear algebraic equation used in the proposed method.

The coefficient matrix of eqn (18) is of order (n+1), symmetric, and nonsingular. The nonsingularity is one of the significant advantages of the proposed method and will be shown in the next section.

3.2 Proof of the nonsingularity of the coefficient matrix[14-15]

The most remarkable characteristic of the proposed method is that nonsingularity is always guaranteed. Let the coefficient matrix of eqn (18) be denoted by C, that is

$$C = \begin{bmatrix} K-\mu M & Mx_i^{(k)} \\ x_i^{(k)T} M & 0 \end{bmatrix} \tag{19}$$

If C is nonsingular when the shift μ becomes an exact eigenvalue, then it will be also nonsingular for a non-close shift. The resulting C^* will be

$$C^* = \begin{bmatrix} K-\lambda_i M & Mx_i \\ x_i^T M & 0 \end{bmatrix} \tag{20}$$

Nonsingularity of the proposed method is, therefore, proved by introducing the new eigenvalue problem of the resulting matrix such as

$$C^* Y = M^* Y D \tag{21}$$

where D and Y are the eigenvalue and the associate eigenvector matrices of the new eigenvalue problem, respectively, and

$$M^* = \begin{bmatrix} M & 0 \\ 0 & 1 \end{bmatrix} \tag{22}$$

$$Y = [y_1, y_2, \ldots, y_{n+1}] \text{ and } D = diag(\gamma_1, \gamma_2, \ldots, \gamma_{n+1}) \tag{23.24}$$

The eigenpairs of the eigenvalue problem eqn (21), y_j and γ_j for $j = 1, 2, \ldots, n+1$, are as follows;

- Eigenvector y_j : $\begin{Bmatrix} x_i \\ -1 \end{Bmatrix}$, $\begin{Bmatrix} x_i \\ 1 \end{Bmatrix}$, $\begin{Bmatrix} x_k \\ 0 \end{Bmatrix}$, $k = 1, 2, \ldots, n,\ k \neq i$
- Eigenvalue γ_j : $-1,\ 1,\ (\lambda_k - \lambda_i)$, $k = 1, 2, \ldots, n,\ k \neq i$

Considering the determinant of eqn (21), the relationship can be obtained as follows;

$$det C^* = det M^* det D = det M \prod_{k=1, k\neq i}^{n} (\lambda_k - \lambda_i) \tag{25}$$

The determinant of C^* is not zero

because of $detM \neq 0$ by definition. The nonsingularity of the coefficient matrix in eqn (18) is shown. That is, the numerical stability of the proposed method is proved analytically. The proposed method, therefore, makes up for the disadvantage that the subspace iteration method with shift has the limitations; no limited regions are needed in the proposed method.

3.3 Operation count and summary of algorithm

Consider the number of Central Processor operations in order to obtain an estimate of the cost required for solving an eigenvalue problem. The actual cost must include, of course, the cost of the Peripheral Processor time. This time is, however, not considered in this investigation since it depends on the system and the programming technique.

Let one operation equal to one multiplication which is nearly always followed by an addition. Assume that the half-bandwidths of K and M are m_K and m_M, respectively. The steps for the subspace iteration method with shift with the operations are summarized in Table 1, and for the proposed method in Table 2.

The number of operations of the subspace iteration method with shift is

$$T_s qn\ (2m_K+4m_M+2q+4)+ n(m_K^2+3m_K+2m_M+2)$$

and that for the proposed method.

$$T_p\ \{qn(2m_K+4m_M+2q+5)+ n(m_{K+1})\}n(m_K^2+3m_K+2m_M+2)$$

Table 1. Operation Count for Subspace Iteration Method with Shift

Operation	Calculation	Number of Operations
Multiplication	$K-\mu M$	$n(m_M+1)$
Factorization	$LDL^T=K-\mu M$	$nm_K(m_K+3)/2$
Iteration	$MX^{(k)}$	$qn(2m_M+1)$
Multiplication	$(K-\mu M)\ \overline{X}^{(k+1)}=MX^{(k)}$	$qn(2m_K+1)$
Solve for $\overline{X}^{(k+1)}$	$\overline{K}^{(k+1)}=\overline{X}^{(k+1)^T}M\ X^{(k)}$	$qn(2m_M+1)$
Multiplication	$M\ \overline{X}^{(k+1)}$	$qn(q+1)/2$
Multiplication	$\overline{M}^{(k+1)}=\overline{X}^{(k+1)^T}M\ \overline{X}^{(k+1)}$	$qn(q+1)/2$
Solve for $Q^{(k+1)}$ & $\Omega^{(k+1)}$	$\overline{K}^{(k+1)}\ Q^{(k+1)}=\overline{M}^{(k+1)}\ Q^{(k+1)}\ \Omega^{(k+1)}$	$0(q^3)$ neglected
Multiplication	$X^{(k+1)}=\overline{X}^{(k+1)}\ Q^{(k+1)}$	nq^2
Subtotal	$\{qn(2m_K+4m_M+2q+4)$	
Sturm Sequence Check Multiplication	$K-\lambda_p M$	$n(m_M+1)$
Factorization	$LDL^T=K-\lambda_p M$	$nm_K(m_K+3)/2$
Total	$T_s qn(2m_K+4m_M+2q+4)+n(m_K^2+3m_K+2m_M+2)$	

Table 2. Operation Count for the Proposed Method

Operation	Calculation	Number of Operations
Iteration		
k=0		
Multiplication	$K-\mu M$	$n(m_M+1)$
Multiplication	$MX^{(0)}$	$qn(2m_M+1)$
Change the last column of $K-\mu M$ into $Mx_i^{(0)}$		neglected
Factoricarion	$LDL^T=F^{(1)}$	$nm_K(m_K+5)/2+n$
k=1,2,...		
Multiplication	$MX^{(k)}$	$qn(2m_M+1)$
Change the last column of $K-\mu M$ into $Mx_i^{(0)}$		neglected
Factoricarion	$LDL^T=F^{(k+1)}$	$n(m_K+1)$
Solve for $\hat{X}^{(k+1)}$	$F^{(k+1)}\ \hat{X}^{(k+1)}=R$	$2qn(m_K+1)$
Multiplication	$\overline{K}^{(k+1)}=\overline{X}^{(k+1)^T}M\ X^{(k)}$	$qn(q+1)/2$
Multiplication	$M\ \overline{X}^{(k+1)}$	$qn(2m_M+1)$
Multiplication	$\overline{M}^{(k+1)}=\overline{X}^{(k+1)^T}M\ \overline{X}^{(k+1)}$	$qn(q+1)/2$
Solve for $Q^{(k+1)}$ & $\Omega^{(k+1)}$	$\overline{K}^{(k+1)}\ Q^{(k+1)}=\overline{M}^{(k+1)}\ Q^{(k+1)}\ \Omega^{(k+1)}$	$0(q^3)$ neglected
Multiplication	$X^{(k+1)}=\overline{X}^{(k+1)}\ Q^{(k+1)}$	nq^2
Subtotal	$qn(2m_K+4m_M+2q+5)+n(m_K+1)$	
Sturm Sequence Check Multiplication	$K-\lambda_p M$	$n(m_M+1)$
Factorization	$LDL^T=K-\lambda_p M$	$nm_K(m_K+3)/2$
Total	$T_p\{qn(2m_K+4m_M+2q+5)+n(m_{K+1})\}n(m_K^2+3m_K+2m_M+2)$	

where

$$F^{(k+1)}=\begin{bmatrix} K-\mu M & Mx_i^{(k)} \\ x_i^{(k)^T} & 0 \end{bmatrix},$$

$$\hat{X}^{(k+1)}=\begin{bmatrix} \overline{X}^{(k+1)} \\ d_i^{(k+1)} \end{bmatrix},$$

$$R=\begin{bmatrix} MX^{(k)} \\ e_i \end{bmatrix} \qquad \text{(see eqn (18))}$$

The proposed method needs more operations per each iteration step, $qn+n(m_K+1)$, than the subspace iteration method with shift. Assume that the ratio is composed of the operation count per iteration of the proposed method(N_p), that of the subspace iteration method with shift(N_s) and the difference of the operation count per iteration for above two methods(N_p-N_s) as follows:

$$ratio=\frac{N_p-N_s}{N_p}$$

$$=\frac{qn+n(m_K+1)}{qn(2m_K+4m_M+2q+5)+n(m_K+1)} \tag{26}$$

Then, if the half-bandwidth of the

stiffness matrix(m_K) is equal to that of the mass matrix(m_M), the above ratio can be approximated as follows:

$$ratio \approx \frac{1}{6q}. \tag{27}$$

This ratio means that the larger the number of the required eigenpairs is, the smaller is the ratio of the difference of the operation count to the system degree of freedom. That is, the number of operations for the aforementioned two methods, the subspace iteration method with shift and the proposed method, is almost the same when the number of eigenpairs to be required is large.

4. Numerical Examples

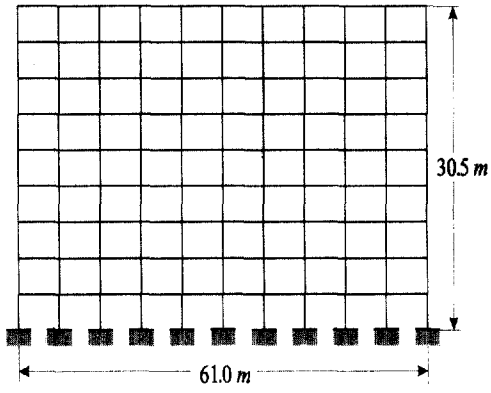
The plane framed structure and the three-dimensional framed structure used by Bathe and Wilson[2] are analyzed to verify the effectiveness of the proposed method. With the predetermined error norm of 10^{-6}, the structures are analyzed by two methods: the subspace iteration method with shift which is not used the limited region(see eqn (11)) and the proposed method, where the error norm is computed by the following equation:

$$error \quad norm = \frac{||(K-\lambda_i^{(k)}M)x_i^{(k)}||_2}{||Kx_i^{(k)}||_2} \tag{28}$$

Even if a shift is on or very close to an eigenvalue, it is shown that the proposed method can obtain the solutions without any singularity. When a shift is not close to an eigenvalue, each convergence rate for calculating the first ten eigenpairs is compared. All runs are executed in the IRIS4D-20-S17 with 10 Mips and 0.9 Mflops.

4.1 Plane framed structure

The first example is a plane framed structure. The geometric configuration and the material properties are shown in Figure 1. The structure discretized using 210 beam elements resulting in system of dynamic equations with a total of 330 degrees of freedom. The consistent mass matrix is used for M. Some results are shown in Table 3 and in Figures 2 to 7. The solution time for two methods are



$A=0.2787\,m^2$, $I=8.6312\times10^{-3}\,m^4$
$E=2.068\times10^{10}\,Pa$, $\rho=5.154\times10^2\,kg/m^3$

Fig 1. Plane framed structure

Table 3. Solution time for the lowest ten eigenvalues of the plane framed structure

	Analysis methods	Solution time, sec (ratio)
Shift = 1.01 λ_3	Subspace iteration method with shift	201.85 (1.00)
	Proposed method	204.78 (1.01)
Shift = λ_3	Subspace iteration method with shift	No solution
	Proposed method	204.35

summarized in Table 3. When a shift is on 1.01 λ_3, the subspace iteration method with shift and the proposed method calculate the required ten eigenpairs. However, when the shift is on λ_3, the subspace iteration method with shift does not find the solutions while the proposed method obtains the solutions. It shows that the iteration procedure for the proposed method can converge without any singularity even if the shift is the same exactly to an eigenvalue, as analytically proved in the article 3.2. This is one of the significant advantages of the proposed method.

For each solution method, the convergence of each eigenpair is depicted in Figures 2 to 7. Figures 2 to 4 show

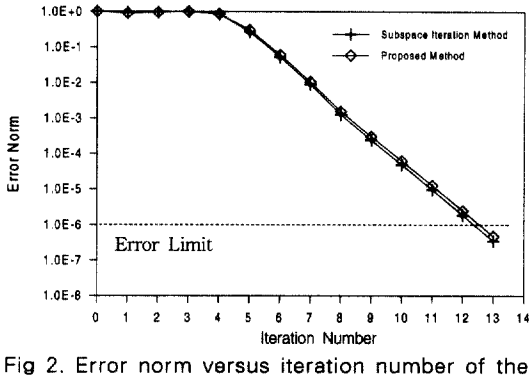


Fig 2. Error norm versus iteration number of the first eigenpair of the plane framed structure in case of shift = 1.01 λ_3.

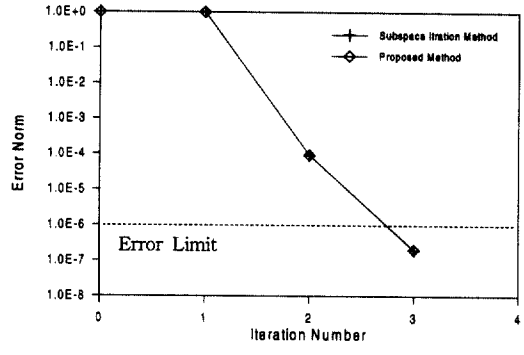


Fig 3. Error norm versus iteration number of the third eigenpair of the plane framed structure in case of shift = 1.01 λ_3.

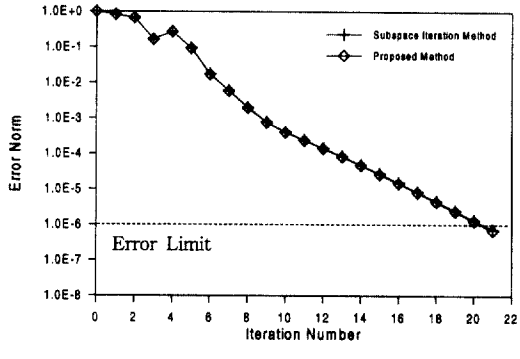


Fig 4. Error norm versus iteration number of the tenth eigenpair of the plane framed structure in case of shift = 1.01 λ_3.

that when the shift is on 1.01 λ_3 the convergence of the proposed method is nearly equal to that of the accelerated subspace iteration method. Figures 5 to 7 show that when the shift is the same

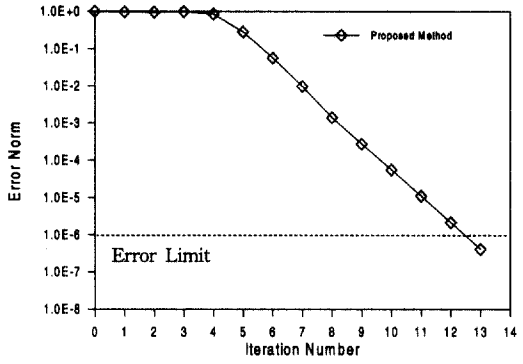


Fig 5. Error norm versus iteration number of the first eigenpair of the plane framed structure in case of shift = λ_3.

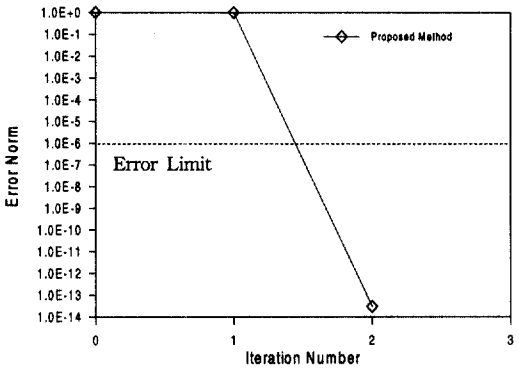


Fig 6. Error norm versus iteration number of the third eigenpair of the plane framed structure in case of shift = λ_3.

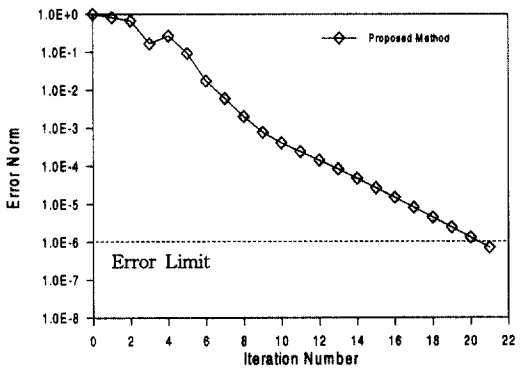


Fig 7. Error norm versus iteration number of the tenth eigenpair of the plane framed structure in case of shift = λ_3.

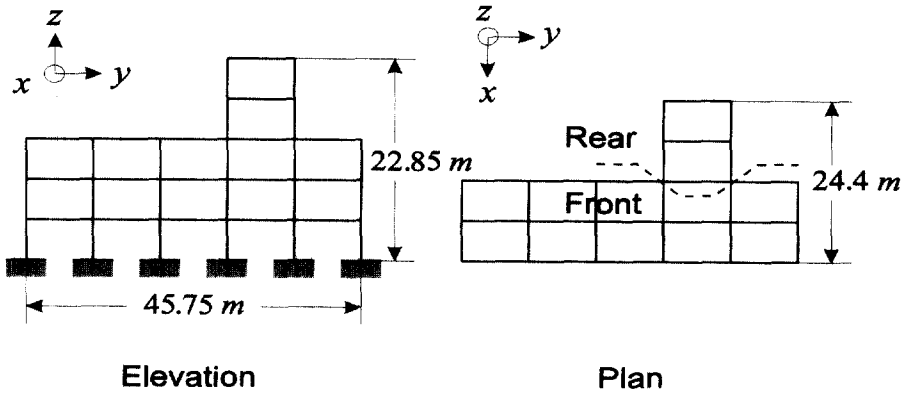
exactly to the third eigenvalue the proposed method converges very well while the subspace iteration method with shift can not converge due to the singularity. As the above results, the proposed method can choose a more exact shift than the subspace iteration method with shift, thus the proposed method may be the more computationally efficient.

4.2 Three-dimensional framed structure

The second example is the three-dimensional framed structure. Figure 8 shows the geometric configuration and material properties. The structure discretized using 100 beam elements resulting in system of dynamic eqns with a total of 468 degrees of freedom. The consistent mass matrix is used for $\boldsymbol{M}$.

Some results are shown in Table 4 and in Figures 9 to 14. The solution time for two methods are summarized in Table 4. When a shift is on $1.01\,\lambda_5$, the subspace iteration method with shift and the proposed method obtain the required ten eigenpairs. However, when the shift is on λ_5, the subspace iteration method with shift does not calculate the solutions while the proposed method finds the solutions.

It shows that the iteration procedure for the proposed method can converge without any singularity even if the shift is the same exactly to an eigenvalue, as analytically proved in the article 3.2.



Column in Front Building : $A = 0.2787\,m^2$, $I = 8.631 \times 10^{-3}\,m^4$

Column in Front Building : $A = 0.3716\,m^2$, $I = 10.787 \times 10^{-3}\,m^4$

All Beams into x-Direction : $A = 0.6096\,m^2$, $I = 6.473 \times 10^{-3}\,m^4$

All Beams into y-Direction : $A = 0.2787\,m^2$, $I = 8.631 \times 10^{-3}\,m^4$

$E = 2.068 \times 10^{10}\,Pa$, $\rho = 5.154 \times 10^2\,kg/m^3$

Fig 8. Three-dimensional framed structure

For each solution method, the convergence of each eigenpair is depicted in Figures 9 to 14. Figures 9 to 11 show that when the shift is on 1.01 λ_5 the convergence of the proposed method is nearly equal to that of the subspace iteration method with shift. Figures 12 to 14 show that when the shift is the same exactly to the fifth eigenvalue the proposed method converges well without any singularity while the subspace iteration method with shift can not converge due to the singularity. As the above results, the

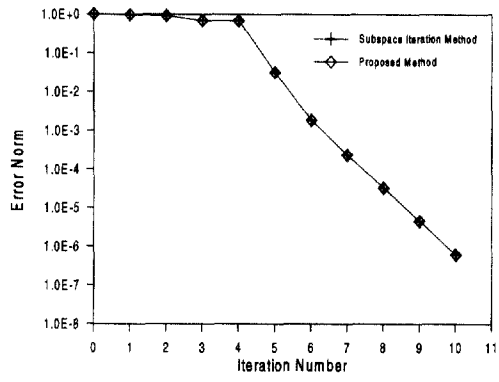


Fig 9. Error norm versus iteration number of the first eigenpair of the three-dimensional framed structure in case of shift = 1.01 λ_5.

Table 4. Solution time for the lowest ten eigenvalues of the 3-dim. framed structure

	Analysis methods	Solution time, sec (ratio)
Shift = 1.01 λ_5	Subspace iteration method with shift	485.36 (1.00)
	Proposed method	492.16 (1.01)
Shift = λ_5	Subspace iteration method with shift	No solution
	Proposed method	491.70

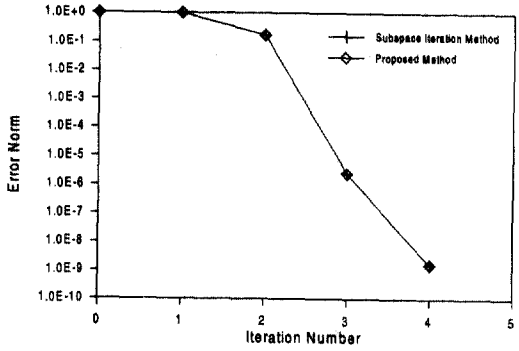


Fig 10. Error norm versus iteration number of the fifth eigenpair of the three-dimensional framed structure in case of shift = 1.01 λ_5.

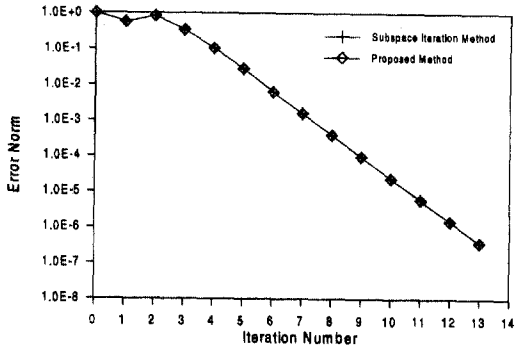


Fig 11. Error norm versus iteration number of the tenth eigenpair of the three-dimensional framed structure in case of shift = 1.01 λ_5.

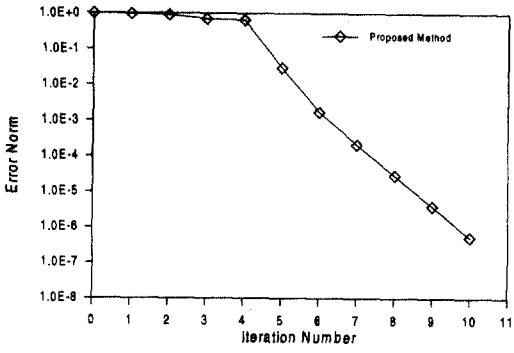


Fig 12. Error norm versus iteration number of the first eigenpair of the three-dimensional framed structure in case of shift = λ_5.

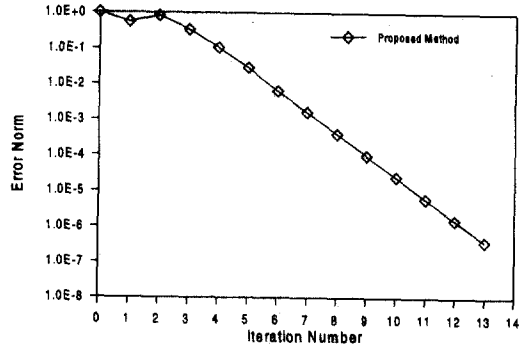


Fig 13. Error norm versus iteration number of the fifth eigenpair of the three-dimensional framed structure in case of shift = λ_5.

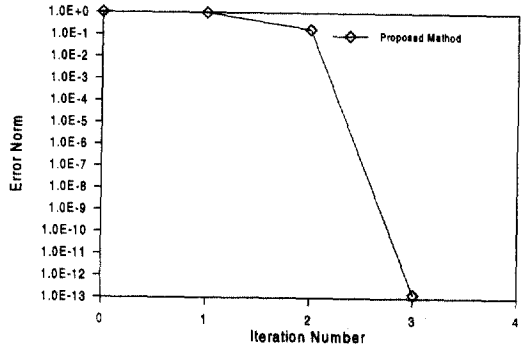


Fig 14. Error norm versus iteration number of the tenth eigenpair of the three-dimensional framed structure in case of shift = λ_5.

proposed method can choose a more aggressive shift than the subspace iteration method with shift.

5. Conclusion

This paper proposes a numerically stable acceleration technique using side conditions for the improvement of the subspace iteration method with shift. The characteristics of the proposed method

identified by the analytical and the numerical results from numerical examples are summarized as follows:

(1) The nonsingularity of the proposed method is always guaranteed, which is proved analytically; even if the shift is an eigenvalue itself, the proposed method can obtain the solutions without any singularity.

(2) The convergence rate of the proposed method is at least equal to that of the subspace iteration method with shift, and the operation counts of the proposed method and the subspace iteration method with shift are almost the same when the number of eigenpairs to be required is large.

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