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A STUDY OF RAMANUJAN $\tau(n)$ NUMBER AND DEDEKIND ETA–FUNCTION

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Abstract In this paper, we consider properties of Dedekind eta-function, modular discrimiant, thata-series and Weierstrass $\wp$-function. We prove the integrablities of $\Delta(\tau)$ and $\eta(\tau)$. Also, we give explicit formulae about $\Delta(\tau)$ and $\tau(n)$.

1. Preliminaries

Let $\Lambda_\tau = \mathbb{Z} + \mathbb{Z}\tau \subset \mathbb{C}$ be a lattice, where $\tau \in \mathbb{C}$ and $\mathrm{Im}\tau > 0$. The *Eisenstein series of weight 2k(for Λ_τ)* is the series $G_{2k}(\Lambda_\tau) = \sum_{\substack{\omega\in\Lambda_\tau \\ \omega\neq 0}} \omega^{-2k}$.

It is standard notation to set $g_2(\Lambda_\tau) = g_2(\tau) = 60G_4$ and $g_3(\Lambda_\tau) = g_3(\tau) = 140G_6$. The *the modular discriminant* is the function $\Delta(\tau) = g_2(\tau)^3 - 27g_3(\tau)^2$.

The *Weierstrass $\wp$-function(relative to Λ)* is defined by the series

$$\wp(z;\Lambda) = \frac{1}{z^2} + \sum_{\substack{\omega\in\Lambda \\ \omega\neq 0}} \left\{ \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right\}.$$

PROPOSITION 1.1. *([5])*

(a)

$$\prod_{\substack{u,v\in\frac{1}{n}\Lambda_\tau/\Lambda_\tau \\ u\neq v}} (\wp(\tau+u)-\wp(\tau+v)) = \pm n^{n^2}\wp'(n\tau)^{n^2-1}\Delta(\tau)^{\frac{(2n^2-3)(n^2-1)}{12}}.$$

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(b)

$$\prod_{\substack{u,v\in\frac{1}{n}\Lambda_\tau/\Lambda_\tau \\ u\not\equiv v(\text{mod }\Lambda_\tau) \\ u,v\not\equiv 0(\text{mod }\Lambda_\tau)}} (\wp(u)-\wp(v)) = \pm n^{-2(n^2-3)}\Delta(\tau)^{\frac{(n^2-1)(n^2-3)}{6}}.$$

Let $p = e^{\pi i\tau}$ and $f(\tau) = \prod_{n=1}^{\infty}(1-p^{2n})(1+p^{2n-1})^2$. Then

$$\Delta(\tau) = 16\pi^{12} f(\tau)^8 f(\tau+1)^8\{f(\tau)^4 - f(\tau+1)^4\}^2. \text{ ([1])}$$

The *Dedekind η-function* $\eta(\tau)$ is defined by the product

$$\eta(\tau) = e^{(2\pi i\tau)/24}\prod_{n=1}^{\infty}(1-q^n) \quad \text{for } \tau\in\mathfrak{h}, q = e^{2\pi i\tau}.$$

Let x and y be relatively prime integers with $y > 0$.

The *Dedekind sum* $s(x,y)$ is defined to be

$$s(x,y) = \sum_{j=1}^{y-1}\frac{j}{y}\left(\frac{jx}{y} - \left[\frac{jx}{y}\right] - \frac{1}{2}\right).$$

(The square bracket denotes the greatest integer function [3].)

PROPOSITION 1.2. *([3])*

(a) *The Dedekind η-function satiesfies the identities*

$$\eta(\tau+1) = e^{\frac{2\pi i}{24}}\eta(\tau), \quad \text{and } \eta\left(-\frac{1}{\tau}\right) = \sqrt{-i\tau}\,\eta(\tau).$$

Here we take the branch of $\sqrt{\ }$ which is positive on the positive real axis.

(b)

$$\Delta(\tau) = (2\pi)^{12}\eta(\tau)^{24}.$$

PROPOSITION 1.3. *(Dedekind,[3]) Let* $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ *with* $c > 0$. *The Dedekind* η*-function satisfies the transformation formula*

$$\eta(\gamma\tau) = e^{2\pi i \Phi(\gamma)/24} \sqrt{-i(c\tau + d)}\eta(\tau),$$

where $\sqrt{\ }$ *is the branch of the square root which is positive on the positive real axis,* $\Phi(\gamma)$ *is given by the formula*

$$\Phi(\gamma) = \frac{1}{c} + \frac{d}{c} - 12s(d, c),$$

and $s(x, y)$ *is Dedekind sum defined above.*

2. The relation of eta-function and other functions

We consider the Dedekind $\eta(\tau)$-function and theta-series.

PROPOSITION 2.1. *([2])*

$$f(\tau) = e^{-\frac{\pi i}{12}} \frac{\eta\left(\frac{\tau+1}{2}\right)^2}{\eta(\tau)}.$$

By Proposition 2.1, we obtain

$$f(\tau) = e^{-\frac{\pi i}{12}} \frac{\eta(\frac{\tau+1}{2})^2}{\eta(\tau)}, \quad f(\tau + 1) = e^{-\frac{\pi i}{12}} \frac{\eta(\frac{\tau+2}{2})^2}{\eta(\tau + 1)}.$$

Let $\rho = e^{\frac{2\pi i}{3}}$ and $-\overline{\rho} = e^{\frac{2\pi i}{6}}$. Taking $f(\tau)$ and $f(\tau + 1)$ replacing to η-equations,

$$\begin{aligned}\Delta(\tau) &= 16\pi^{12} f(\tau)^8 f(\tau + 1)^8 \{f(\tau)^4 - f(\tau + 1)^4\}^2 \\ &= 16\pi^{12}(\rho) \frac{\eta(\frac{\tau+1}{2})^{16} \eta(\frac{\tau+2}{2})^{16}}{\eta(\tau)^{24}} \left((-\overline{\rho})\eta(\frac{\tau + 1}{2})^8 - \eta(\frac{\tau + 2}{2})^8 \right)^2.\end{aligned}$$

PROPOSITION 2.2. *([1]) Let $\Delta(\tau)=\{\epsilon\Omega(\tau)\}^3$, where $\epsilon=1,\rho$ or $\overline{\rho}$.*

(a) $$\eta(\tau)^{48}=\frac{1}{256}\Big[2\eta(\tfrac{\tau}{2})^{24}\eta(\tfrac{\tau+1}{2})^{24} +\eta(\tfrac{\tau}{2})^{16}\eta(\tfrac{\tau+1}{2})^{16}\left(\rho\eta(\tfrac{\tau+1}{2})^{16}+\overline{\rho}\eta(\tfrac{\tau}{2})^{16}\right)\Big].$$

(b) $$\Omega(\tau)^6=\frac{1}{256}\Omega(\tfrac{\tau}{2})^2\Omega(\tfrac{\tau+1}{2})^2[\Omega(\tfrac{\tau+1}{2})\left(\Omega(\tfrac{\tau}{2})+\rho\Omega(\tfrac{\tau+1}{2})\right) +\Omega(\tfrac{\tau}{2})\left(\Omega(\tfrac{\tau+1}{2})+\overline{\rho}\Omega(\tfrac{\tau}{2})\right)].$$

(c) $$\eta(\tau)^{24}=\frac{1}{256}f(\tau)^8f(\tau+1)^8(f(\tau)^4-f(\tau+1)^8)^2.$$

THEOREM 2.3.

$$\Delta(\tau)^6=\frac{1}{16777216}\Big[393216\Delta(\tau)^4\Delta(\frac{\tau}{2})\Delta(\frac{\tau+1}{2}) - 2304\Delta(\tau)^2\Delta(\frac{\tau}{2})^2\Delta(\frac{\tau+1}{2})^2+2\Delta(\frac{\tau}{2})^3\Delta(\frac{\tau+1}{2})^3 +\Delta(\frac{\tau}{2})^4\Delta(\frac{\tau+1}{2})^2+\Delta(\frac{\tau}{2})^2\Delta(\frac{\tau+1}{2})^4\Big].$$

Proof. By Proposition 2.2,

$$256\Delta(\tau)^2=2\Delta(\frac{\tau}{2})\Delta(\frac{\tau+1}{2})+\Delta(\frac{\tau}{2})^{\frac{2}{3}}\Delta(\frac{\tau+1}{2})^{\frac{2}{3}} \cdot\left(\rho\Delta(\frac{\tau+1}{2})^{\frac{2}{3}}+\overline{\rho}\Delta(\frac{\tau}{2})^{\frac{2}{3}}\right).$$

Thus we have
$$\frac{256\Delta(\tau)^2-2\Delta(\frac{\tau}{2})\Delta(\frac{\tau+1}{2})}{\Delta(\frac{\tau}{2})^{\frac{2}{3}}\Delta(\frac{\tau+1}{2})^{\frac{2}{3}}}=\left(\rho\Delta(\tfrac{\tau+1}{2})^{\frac{2}{3}}+\overline{\rho}\Delta(\tfrac{\tau}{2})^{\frac{2}{3}}\right).$$
Cubing both sides of the above equation,
$$\left(\frac{256\Delta(\tau)^2-2\Delta(\frac{\tau}{2})\Delta(\frac{\tau+1}{2})}{\Delta(\frac{\tau}{2})^{\frac{2}{3}}\Delta(\frac{\tau+1}{2})^{\frac{2}{3}}}\right)^3=\left(\rho\Delta(\tfrac{\tau+1}{2})^{\frac{2}{3}}+\overline{\rho}\Delta(\tfrac{\tau}{2})^{\frac{2}{3}}\right)^3$$

$$= \Delta(\tfrac{\tau}{2})^2 + \Delta(\tfrac{\tau+1}{2})^2 + 3\Delta(\tfrac{\tau}{2})^{\frac{2}{3}}\Delta(\tfrac{\tau+1}{2})^{\frac{2}{3}}\left(\rho\Delta(\tfrac{\tau+1}{2})^{\frac{2}{3}} + \overline{\rho}\Delta(\tfrac{\tau}{2})^{\frac{2}{3}}\right).$$

Multiplying $\Delta(\frac{\tau}{2})^2\Delta(\frac{\tau+1}{2})^2$ on both sides of the above equation,

$$\begin{aligned}&16777216\Delta(\tau)^6 - 393216\Delta(\tau)^4\Delta(\tfrac{\tau}{2})\Delta(\tfrac{\tau+1}{2})\\ &+3072\Delta(\tau)^2\Delta(\tfrac{\tau}{2})^2\Delta(\tfrac{\tau+1}{2})^2 - 8\Delta(\tfrac{\tau}{2})^3\Delta(\tfrac{\tau+1}{2})^3\\ &= \Delta(\tfrac{\tau}{2})^4\Delta(\tfrac{\tau+1}{2})^2 + \Delta(\tfrac{\tau}{2})^2\Delta(\tfrac{\tau+1}{2})^4\\ &+3\Delta(\tfrac{\tau}{2})^2\Delta(\tfrac{\tau+1}{2})^2 \cdot \Delta(\tfrac{\tau}{2})^{\frac{2}{3}}\Delta(\tfrac{\tau+1}{2})^{\frac{2}{3}}\left(\rho\Delta(\tfrac{\tau+1}{2})^{\frac{2}{3}} + \overline{\rho}\Delta(\tfrac{\tau}{2})^{\frac{2}{3}}\right)\\ &= \Delta(\tfrac{\tau}{2})^4\Delta(\tfrac{\tau+1}{2})^2 + \Delta(\tfrac{\tau}{2})^2\Delta(\tfrac{\tau+1}{2})^4\\ &+3\Delta(\tfrac{\tau}{2})^2\Delta(\tfrac{\tau+1}{2})^2 \cdot \left(256\Delta(\tau)^2 - 2\Delta(\tfrac{\tau}{2})\Delta(\tfrac{\tau+1}{2})\right).\end{aligned}$$

Thus,

$$\begin{aligned}&16777216\Delta(\tau)^6 - 393216\Delta(\tau)^4\Delta(\tfrac{\tau}{2})\Delta(\tfrac{\tau+1}{2})\\ &+2304\Delta(\tau)^2\Delta(\tfrac{\tau}{2})^2\Delta(\tfrac{\tau+1}{2})^2 - 2\Delta(\tfrac{\tau}{2})^3\Delta(\tfrac{\tau+1}{2})^3\\ &= \Delta(\tfrac{\tau}{2})^4\Delta(\tfrac{\tau+1}{2})^2 + \Delta(\tfrac{\tau}{2})^2\Delta(\tfrac{\tau+1}{2})^4.\end{aligned}$$

Thus, we have the conclusion.

THEOREM 2.4.

(a) *$\Delta(\tau)$ is a integral over $\mathbb{Q}[\Delta(\frac{\tau}{2}), \Delta(\frac{\tau+1}{2})]$.*
(b) *$\eta(\tau)$ is a integral over $\mathbb{Q}[\eta(\frac{\tau}{2}), \eta(\frac{\tau+1}{2})]$.*
(c) *If $n \geq 2$ is any integer, $\Delta(\tau)$ is a integral over $\mathbb{Q}[\wp(\frac{1}{n}), \wp(\frac{\tau}{n}), \wp(\frac{\tau+1}{n}), \cdots, \wp(\frac{\tau+n-1}{n})]$.*

Proof. (a) By Theorem 2.3.
(b) By Proposition 2.2.(a).
(c)

$$\Delta(\tau)^{\frac{(n^2-1)(n^2-3)}{6}} = \pm n^{-2(n^2-3)} \prod_{\substack{u,v\in\frac{1}{n}\Lambda_\tau/\Lambda_\tau\\ u\not\equiv v (\mathrm{mod}\ \Lambda_\tau)\\ u,v\not\equiv 0(\mathrm{mod}\ \Lambda_\tau)}} (\wp(u) - \wp(v)).$$

But for $u, v \in \dfrac{1}{n}\Lambda_\tau/\Lambda_\tau$, $\wp(u) - \wp(v) \in \mathbb{Q}[\wp(\frac{1}{n}), \wp(\frac{\tau}{n}), \cdots, \wp(\frac{\tau+n-1}{n})]$.

More explicitly, when $n = 2$,

$$\prod_{\substack{u,v\in\frac{1}{2}\Lambda_\tau/\Lambda_\tau \\ u\not\equiv v(\text{mod }\Lambda_\tau) \\ u,v\not\equiv 0(\text{mod }\Lambda_\tau)}} (\wp(u)-\wp(v)) = \pm 2^{-2(2^2-3)}\Delta(\tau)^{\frac{(2^2-1)(2^2-3)}{6}}$$

$$= \pm 2^{-2}\Delta(\tau)^{\frac{1}{2}}$$

and

$$\prod_{\substack{u,v\in\frac{1}{2}\Lambda_\tau/\Lambda_\tau \\ u\not\equiv v(\text{mod }\Lambda_\tau) \\ u,v\not\equiv 0(\text{mod }\Lambda_\tau)}} (\wp(u)-\wp(v)) = \left(\wp(\frac{\tau+1}{2})-\wp(\frac{\tau}{2})\right)^2\left(\wp(\frac{1}{2})-\wp(\frac{\tau}{2})\right)^2$$

$$\cdot\left(\wp(\frac{\tau+1}{2})-\wp(\frac{1}{2})\right)^2.$$

Therefore,

$$\Delta(\tau) = 2^4\left(\wp(\frac{\tau+1}{2})-\wp(\frac{\tau}{2})\right)^4\left(\wp(\frac{1}{2})-\wp(\frac{\tau}{2})\right)^4\left(\wp(\frac{\tau+1}{2})-\wp(\frac{1}{2})\right)^4.$$

Similarly, when $n=3$,

$$\Delta(\tau)^8 = \pm 3^{12}\prod_{\substack{u,v\in\frac{1}{3}\Lambda_\tau/\Lambda_\tau \\ u\not\equiv v(\text{mod }\Lambda_\tau) \\ u,v\not\equiv 0(\text{mod }\Lambda_\tau)}} (\wp(u)-\wp(v))$$

$$= \pm 3^{12}\left(\wp(\frac{1}{3})-\wp(\frac{\tau}{3})\right)^2\left(\wp(\frac{1}{3})-\wp(\frac{\tau+1}{3})\right)^2\left(\wp(\frac{1}{3})-\wp(\frac{\tau+2}{3})\right)^2$$

$$\cdot\left(\wp(\frac{\tau}{3})-\wp(\frac{\tau+1}{3})\right)^2\left(\wp(\frac{\tau}{3})-\wp(\frac{\tau+2}{3})\right)^2\left(\wp(\frac{\tau+1}{3})-\wp(\frac{\tau+2}{3})\right)^2$$

3. g_2 and Ramanujan number $\tau(n)$

By definition,

$$\Delta(\tau) = g_2(\tau)^3 - 27g_3(\tau)^2$$

is the discriminant of the cubic polynomial

$$4\wp^3 - g_2(\tau)\wp - g_3(\tau) = (\wp-e_1)(\wp-e_2)(\wp-e_3).$$

But we know the roots of this polynomial from [4. VI.3.6.], namely

$$e_1 = \wp\left(\frac{1}{2}\right), \quad e_2 = \wp\left(\frac{\tau}{2}\right), \quad e_3 = \wp\left(\frac{\tau+1}{2}\right).$$

Since $\wp(\frac{1}{2}), \wp(\frac{\tau}{2}), \wp(\frac{\tau+1}{2})$ are three distinct roots of the equation

$$4\wp(\tau)^3 - g_2\wp(\tau) - g_3 = 0,$$

we have

$$\wp(\frac{1}{2}) + \wp(\frac{\tau}{2}) + \wp(\frac{\tau+1}{2}) = 0,$$

$$\wp(\frac{1}{2})\wp(\frac{\tau}{2}) + \wp(\frac{1}{2})\wp(\frac{\tau+1}{2}) + \wp(\frac{\tau}{2})\wp(\frac{\tau+1}{2}) = -\frac{g_2}{4},$$

$$\wp(\frac{1}{2})\wp(\frac{\tau}{2})\wp(\frac{\tau+1}{2}) = \frac{g_3}{4},$$

while $\Delta(\tau) = 16(\wp(\frac{1}{2})-\wp(\frac{\tau}{2}))^2(\wp(\frac{\tau+1}{2})-\wp(\frac{\tau}{2}))^2(\wp(\frac{1}{2})-\wp(\frac{\tau+1}{2}))^2$. Now

$$\begin{aligned}\wp(\frac{1}{2})^2 + \wp(\frac{\tau}{2})^2 + \wp(\frac{\tau+1}{2})^2 &= 2\left(\wp(\frac{1}{2}) + \wp(\frac{\tau}{2}) + \wp(\frac{\tau+1}{2})\right)^2 \\ &\quad - 2\left(\wp(\frac{1}{2})\wp(\frac{\tau}{2}) + \wp(\frac{\tau}{2})\wp(\frac{\tau+1}{2}) + \wp(\frac{1}{2})\wp(\frac{\tau+1}{2})\right) \\ &= -2\left(-\frac{g_2}{4}\right) = \frac{g_2}{2},\end{aligned}$$

hence $\left(\wp(\frac{1}{2}) - \wp(\frac{\tau}{2})\right)^2 + \left(\wp(\frac{\tau}{2}) - \wp(\frac{\tau+1}{2})\right)^2 + \left(\wp(\frac{1}{2}) - \wp(\frac{\tau+1}{2})\right)^2$
$= 2(\wp(\frac{1}{2})^2 + \wp(\frac{\tau}{2})^2 + \wp(\frac{\tau+1}{2})^2) - 2(\wp(\frac{1}{2})\wp(\frac{\tau}{2}) + \wp(\frac{\tau}{2})\wp(\frac{\tau+1}{2}) + \wp(\frac{1}{2})\wp(\frac{\tau+1}{2}))$
$= 2 \cdot \frac{g_2}{2} - 2\left(-\frac{g_2}{4}\right) = \frac{3}{2}g_2$, so

$$g_2 = \frac{2}{3}\left(\wp(\frac{1}{2}) - \wp(\frac{\tau}{2})\right)^2 + \left(\wp(\frac{1}{2}) - \wp(\frac{\tau+1}{2})\right)^2 + \left(\wp(\frac{\tau+1}{2}) - \wp(\frac{\tau}{2})\right)^2.$$

Thus

$$\begin{aligned}J(\tau) &= \frac{g_2(\tau)^3}{\Delta(\tau)} \\ &= \frac{\left(\frac{2}{3}\left(\wp(\frac{1}{2}) - \wp(\frac{\tau}{2})\right)^2 + \left(\wp(\frac{1}{2}) - \wp(\frac{\tau+1}{2})\right)^2 + \left(\wp(\frac{\tau+1}{2}) - \wp(\frac{\tau}{2})\right)^2\right)^3}{16(\wp(\frac{1}{2}) - \wp(\frac{\tau}{2}))^2(\wp(\frac{\tau+1}{2}) - \wp(\frac{\tau}{2}))^2(\wp(\frac{1}{2}) - \wp(\frac{\tau+1}{2}))^2} \\ &= \frac{1}{54}\frac{\left(\left(\wp(\frac{1}{2}) - \wp(\frac{\tau}{2})\right)^2 + \left(\wp(\frac{1}{2}) - \wp(\frac{\tau+1}{2})\right)^2 + \left(\wp(\frac{\tau+1}{2}) - \wp(\frac{\tau}{2})\right)^2\right)^3}{(\wp(\frac{1}{2}) - \wp(\frac{\tau}{2}))^2(\wp(\frac{\tau+1}{2}) - \wp(\frac{\tau}{2}))^2(\wp(\frac{1}{2}) - \wp(\frac{\tau+1}{2}))^2}.\end{aligned}$$

Taking the Weierstrass $\wp$-function replacing to $f(\tau)$ and $f(\tau+1)$-functions,

$$J(\tau)=\frac{1}{54}\frac{\left(\pi^4 f(\tau)^8+\pi^4 f(\tau+1)^8+\pi^4\left(f(\tau)^4-f(\tau+1)^4\right)^2\right)^3}{\pi^{12}f(\tau)^8 f(\tau+1)^8\left(f(\tau)^4-f(\tau+1)^4\right)^2}$$
$$=\frac{1}{54}\frac{\left(f(\tau)^8+f(\tau+1)^8+\left(f(\tau)^4-f(\tau+1)^4\right)^2\right)^3}{f(\tau)^8 f(\tau+1)^8\left(f(\tau)^4-f(\tau+1)^4\right)^2}.$$

By Proposition 2.1, we obtain

$$f(\tau)=e^{-\frac{\pi i}{12}}\frac{\eta(\frac{\tau+1}{2})^2}{\eta(\tau)},\quad f(\tau+1)=e^{-\frac{\pi i}{12}}\frac{\eta(\frac{\tau+2}{2})^2}{\eta(\tau+1)}.$$

Taking $f(\tau)$ and $f(\tau+1)$ replacing to η-equations,

$$J(\tau)=\frac{4}{27}\frac{\left(\left(\eta(\frac{\tau+1}{2})^8+\eta(\frac{\tau}{2})^8\right)^2-3\eta(\frac{\tau+1}{2})^8\eta(\frac{\tau}{2})^8\right)^3}{\eta(\frac{\tau+1}{2})^{16}\eta(\frac{\tau}{2})^{16}\left(\rho\eta(\frac{\tau+1}{2})^8+\overline{\rho}\eta(\frac{\tau}{2})^8\right)^2}.$$

Thus we have the theorem.

THEOREM 3.1. *$J(\tau)$ or $g_2(\tau)$ is zero if and only if $\theta_3(\tau)^4=\omega\theta_3(\tau+1)^4$ if and only if $\eta(\frac{\tau+1}{2})^8=\omega\eta(\frac{\tau}{2})^8$ with $\omega^2-\omega+1=0$.*

The modular discriminant has the Fourier expansion

$$\Delta(\tau)=(2\pi)^{12}\sum_{n\geq 1}\tau(n)q^n,$$

where $\tau(1)=1$ and $\tau(n)\in\mathbb{Z}$ for all n. The arithmetic function $n\mapsto\tau(n)$ is called the *Ramanujan τ-function.*

By Theorem 2.3, we obtain the long equation of Ramanujan number $\tau(n)$,

$$\left(\sum_{n\geq 1}\tau(n)p^{2n}\right)^6=\frac{1}{16777216}$$
$$\cdot\Bigg[393216\left(\sum_{n\geq 1}\tau(n)p^{2n}\right)^4\left(\sum_{n\geq 1}\tau(n)p^n\right)\left(\sum_{n\geq 1}\tau(n)(-p)^n\right)$$

$$-2304\left(\sum_{n\geq 1}\tau(n)p^{2n}\right)^2\left(\sum_{n\geq 1}\tau(n)p^{n}\right)^2\left(\sum_{n\geq 1}\tau(n)(-p)^{n}\right)^2$$
$$+2\left(\sum_{n\geq 1}\tau(n)p^{n}\right)^3\left(\sum_{n\geq 1}\tau(n)(-p)^{n}\right)^3$$
$$+\left(\sum_{n\geq 1}\tau(n)p^{n}\right)^4\left(\sum_{n\geq 1}\tau(n)(-p)^{n}\right)^2$$
$$+\left(\sum_{n\geq 1}\tau(n)p^{n}\right)^2\left(\sum_{n\geq 1}\tau(n)(-p)^{n}\right)^4\Bigg].$$

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