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ON UNIFORMITIES OF BCI–ALGEBRAS

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Abstract In this paper, we construct the uniformity of a BCI-algebra.

1. Introduction

By a *BCI-algebra* we mean an algebra $(X; *, 0)$ of type (2,0) satisfying the following axioms:

(I) $(x * y) * (x * z) \leq (z * y)$,
(II) $x * (x * y) \leq y$,
(III) $x \leq x$,
(IV) $x \leq y$ and $y \leq x$ implies $x = y$,

for all $x, y, z \in X$. We can defined a partial ordering $\leq$ by $x \leq y$ if and only if $x * y = 0$.

In what follows, X would mean a BCI-algebra unless otherwise specified. We first recall some definitions and properties.

Definition 1.1. ([5]). Let A be a nonempty subset of X. Then A is called to be an *ideal* of X if, for all $x, y \in X$,

(i) $0 \in A$,
(ii) $x * y \in A$ and $y \in A$ imply $x \in A$.

Definition 1.2. ([5]). Let A be a subset of X. Define the ideal of X *generated* by A, denote $< A >$, to be the intersection of all ideals of X which contain A.

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LEMMA 1.3. *([5, 6]). Let A be a subset of X and there exists some $x \in A$ satisfying $x \geq 0$. Then $< A >$ can be described as the set of all $y \in X$ such that $(\cdots((y * a_1) * a_2) * \cdots) * a_n = 0$ for some $a_1, a_2, \cdots, a_n \in A$.*

DEFINITION 1.4. ([7]). Let M be any nonempty set and let U and V be any subsets of $M \times M$. Define

$U \circ V = \{(x, y) \in M \times M|$ for some $z \in M, (x, z) \in U$ and $(z, y) \in V\}$,

$U^{-1} = \{(x, y) \in M \times M|(y, x) \in U\}$,

$\triangle = \{(x, x) \in M \times M|x \in M\}$.

By a *uniformity* on M we mean a nonempty collection K of subsets of $M \times M$ which satisfies the following conditions:

(U_1) $\triangle \subset U$ for any $U \in K$,

(U_2) if $U \in K$, then $U^{-1} \in K$,

(U_3) if $U \in K$, then there exists a $V \in K$ such that $V \circ V \subset U$,

(U_4) if $U, V \in K$, then $U \cap V \in K$,

(U_5) if $U \in K$ and $U \subset V \subset M \times M$, then $V \in K$.

The pair (M, K) is called a *uniform structure.*

LEMMA 1.5. *([2]). For any $x, y \in X$ and any positive integer n, we have $0 * (x * y)^n = (0 * x^n) * (0 * y^n)$, where $x * y^n$ denote the element $(\cdots((x * y) * y) * \cdots) * y$ (y occurs n times).*

In this paper, we construct the uniformity of BCI-algebras which is a generalization of one in the sense of Alo and Deeba ([1]).

2. Main Results

THEOREM 2.1. *Let A be an ideal of X. For every natural number n, we define*

$U_A = \{(x, y) \in X \times X|0 * (x * y)^n \in A$ *and* $0 * (y * x)^n \in A\}$

and let

$K^* = \{U_A|$ *A is an ideal of X*$\}$.

Then K^* satisfies the conditions $(U_1) - (U_4)$.

Proof. Let $(x, x) \in \triangle$. Since $0 * (x * x)^n = 0 * 0^n = 0 \in A$ for any ideal A, it follows that $(x, x) \in U_A$ for every $U_A \in K^*$, which proves that (U_1) holds.

For any $U_A \in K^*$, $(x, y) \in U_A$ if and only if $0 * (x * y)^n \in A$ and $0 * (y * x)^n \in A$ if and only if $(y, x) \in {U_A}^{-1}$ if and only if $(x, y) \in {U_A}^{-1}$. Hence ${U_A}^{-1} = U_A \in K^*$, which is (U_2).

Assume that $U_A \in K^*$. Let $\mathcal{A} = \{I_\alpha | \alpha \in \Lambda\}$ be a collection of ideals of X which is contained in A. $\mathcal{B}$ is a subcollection of $\mathcal{A}$ such that at least one member of $\mathcal{B}$ contains an element $x \geq 0$. Let J be the ideal generated by $\bigcup_\beta I_\beta$, where $I_\beta \in \mathcal{B}$. Now, we show that $A' = U_J$ is such that $U_J \circ U_J \subset U_A$. Let $(x, y) \in U_J \circ U_J$. The definition of $U_J \circ U_J$ implies that for some $z \in X$, $(x, z) \in U_J$ and $(z, y) \in U_J$. That is, $0 * (x * z)^n, 0 * (z * x)^n \in J$ and $0 * (z * y)^n, 0 * (y * z)^n \in J$. Thus we have

$$\begin{aligned}
&(0 * (y * x)^n) * (0 * (y * z)^n) \\
&= ((0 * y^n) * (0 * x^n)) * ((0 * y^n) * (0 * z^n)) \\
&\leq (0 * z^n) * (0 * x^n) \\
&= 0 * (z * x)^n.
\end{aligned}$$

Since J is an ideal, $0 * (y * x)^n \in J$. Similarly we can show that $0 * (x * y)^n \in J$. Since J is the minimal ideal containing $\bigcup_\beta I_\beta$ and since $\bigcup_\beta I_\beta \subset A$, it follows that $J \subset A$. Hence $0*(x*y)^n, 0*(y*x)^n \in A$. Thus $(x, y) \in U_A$ and so $U_J \circ U_J \subset U_A$, which is (U_3).

Finally we prove (U_4). This will follow from the observation that $U_A \cap U_{A'} = U_{A \cap A'}$. Let $(x, y) \in U_A \cap U_{A'}$. Then $(x, y) \in U_A$ and $(x, y) \in U_{A'}$. This implies that

$$0 * (x * y)^n, 0 * (y * x)^n \in A \quad \text{and} \quad 0 * (x * y)^n, 0 * (y * x)^n \in A'.$$

Hence $0 * (x * y)^n, 0 * (y * x)^n \in A \cap A'$ and this implies that $(x, y) \in U_{A \cap A'}$. So $U_A \cap U_{A'} \subset U_{A \cap A'}$. Likewise we can show that $U_{A \cap A'} \subset U_A \cap U_{A'}$. Thus $U_A \cap U_{A'} = U_{A \cap A'}$ and this proves requirement (U_4).

THEOREM 2.2. *Let $K = \{U \subset X \times X | U \supset U_A$ for some $U_A \in K^*\}$ where $K^* = \{U_A | A$ is an ideal of $X\}$. Then K satisfies a uniformity on X. The pair (X, K) is a uniform structure.*

Proof. Using Theorem 2.1, we can show that K satisfies the conditions $(U_1) - (U_4)$. To prove (U_5), let $U \in K$ and $U \subset V \subset$

$X \times X$. Then there exists a $U_A \in K^*$ such that $U_A \subset U \subset V$, which implies that $V \in K$. This completes the proof.

DEFINITION 2.3. For $x \in X$ and $U \in K$, we define

$$U[x] = \{y \in X | (x, y) \in U\}.$$

THEOREM 2.4. *For each $x \in X$, the collection $\mathcal{U}_x = \{U[x] | U \in K\}$ forms a neighborhood base at x, making X a topological space.*

Proof. First note that $x \in U[x]$ for each x. Second,

$$U_1[x] \cap U_2[x] = (U_1 \cap U_2)[x],$$

which means that the intersection of neighborhoods is a neighborhood. Finally, if $U[x] \in \mathcal{U}_x$ then by (U_3) there exists a $E \in K$ such that $E \circ E \subset U$. Then for any $y \in E[x], E[y] \subset U[x]$, so this propery of neighborhoods is satisfied.

References

1. R. A. Alo and E. Y. Deeba, *A note on uniformities of a BCK-algebra*, Math. Japon. **30** (1985), 237-240.
2. W. Huang, *Nil-radical in BCI-algebras*, Math. Japon. **37** (1992), 363-366.
3. K. Iséki, *On a quasi-uniformity on BCK-algebras*, Math. Seminar Notes **4** (1976), 225-226.
4. K. Iséki, *Introduction of a quasi-uniformity on BCK-algebras*, Math. Seminar Notes **4** (1976), 229-230.
5. K. Iséki and S. Tanaka, *An introduction to the theory of BCK-algebras*, Math. Japon. **23** (1978), 1-26.
6. S. M. Wei, Y. B. Jun and E. H. Roh, *A note on ideals in BCI-algebras*, Math. Japon. **42** (1995), 69-73.
7. S. Willard, *General Topology*, Addison-Wesley Publishing Co. (1970).