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UNIQUENESS AND EXISTENCE OF THE SOLUTION FOR CERTAIN SCALAR SEMILINEAR ELLIPTIC PROBLEM

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Abstract We show by using some properties of an elliptic integral that certain scalar semilinear elliptic problem has a unique solution.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$ with sufficiently smooth boundary $\partial\Omega$. Consider the following steady-state problem

$$(1)\qquad \begin{cases} -\Delta u = f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

of the associated semilinear parabolic PDE

$$(2)\qquad \begin{cases} u_t - \Delta u = f(u) & \text{in } \Omega, \\ u(x,0) = \phi(x) & \text{in } \Omega, \\ u(x,t) = 0 & \text{on } \partial\Omega \times (0,\infty). \end{cases}$$

Lots of informations and data for the equation (1) are necessary to study a blowup phenomenon of the equation (2).

Berbernes and Eberly [2] has shown details of the problem (1) with $f(u) = \delta e^u$ (δ a positive parameter), so-called Gelfand problem, and its perturbation (Gelfand-like problem).

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Chafee and Infante [3] has described and investigated the bifurcation equation for the problem (1) with $f(u) = \lambda g(u)$, where the nonlinear function $g(u)$ satisfies certain conditions.

Also the problem (1) with $f(u) = u^p, u^p - u$ or similar types were studied by many authors (for example, see Ambrosetti and Prodi [1], Korman and Ouyang [4,5], Ni [6,7], Smoller [8], Smoller and Wasserman [9], and references therein).

We may ask the following question. What if $f(u)$ is the multiplication of the first (exponential function) and the second (polynomial function)? The purpose of this paper here is to study the existence and the uniqueness of the positive solution of the problem (1) with $f(u) = ue^u$ and $\Omega = B(0,1) \subset \mathbb{R}^N$, the unit ball centered at the origin. In particular we restrict our interests to the radially symmetric case, with $N = 1$ so that $\Omega = B(0,1) = (-1,1)$.

2. Main Threom and Its Proof

Thus our problem (1) can be rewritten as

$$
\begin{aligned}
&-\frac{d^2u}{dx^2} = ue^u, \quad 0 < x < 1, \\
&u'(0) = 0 = u(1)(u' = \frac{du}{dx}).
\end{aligned}
\tag{3}
$$

Here we require that $u(x)$ is positive for $0 \le x < 1$ and $u'(x)$ is nonpositive for $0 \le x \le 1$. From (3), multiplying it by $\frac{du}{dx}$ and then an integration gives us the following.

$$
H(u,v) \equiv: \frac{v^2}{2} + G(u) = h \tag{4}
$$

where $v = \frac{du}{dx}$ and $G(u) = e^u(u-1) + 1$.

$G : [0, \infty) \to [0, \infty)$ has an inverse function on $(0, \infty)$ with $G'(u) = e^u u > 0$ for $u > 0$.

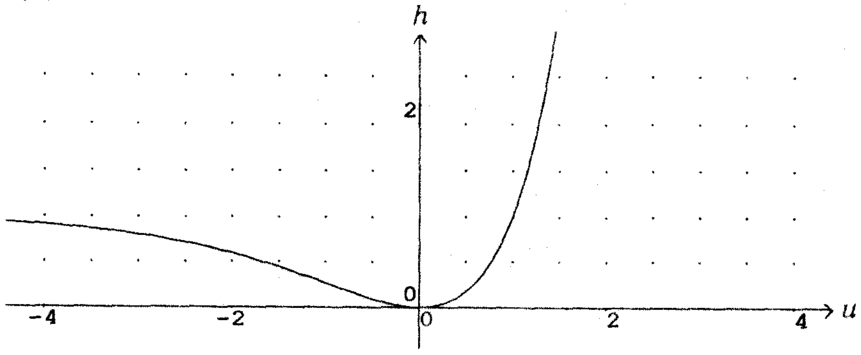

Figure 1. The Graph of $h = G(u)$.

To rewrite (3) as a system of first order equations, setting $u' = v$, we obtain

$$\begin{cases} u' = & v \\ v' = & -ue^u. \end{cases} \tag{5}$$

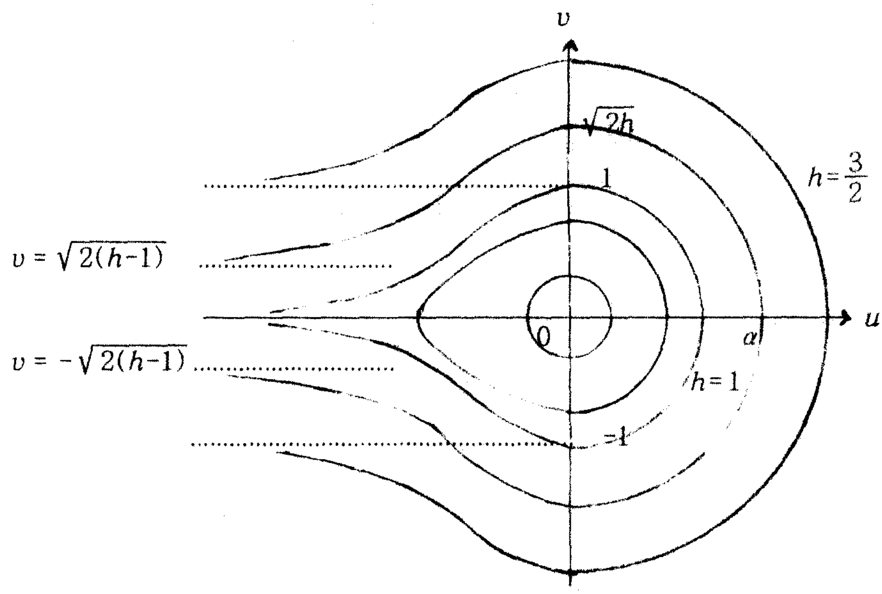

Figure 2. Phase portrait for (5) or Level curves of $\frac{v^2}{2} + G(u) = h$.

Note that $H(0,0) = 0$ and $(0,0)$ is a center. We can show that if $h \leq 1$, v approaches to $\pm\sqrt{2(h-1)}$ as x tends to $\pm\infty$, respectively (so, as u tends to $-\infty$). Also the curve is a periodic orbit for $0 < h < 1$.

MAIN THEOREM. *The equation* (3) *has a unique positive solution.*

Let $\alpha = G^{-1}(h)$, that is , $h = G(\alpha)$.

$$\frac{du}{dx} = v = -\sqrt{2(G(\alpha) - G(u))}, (\text{ since } \frac{du}{dx} \leq 0).$$

From the equation (3),

$$u(0) = \alpha = G^{-1}(h), \quad v(0) = 0, \quad u(1) = 0, \quad v(1) = -\sqrt{2h}$$

$$\int_{u(0)}^{u(1)} \frac{du}{-\sqrt{2(G(\alpha) - G(u))}} = \int_0^1 dr = 1$$

To prove theorem, it is enough to show that there exits a unique $\alpha \in (0, \infty)$ such that

$$\frac{1}{\sqrt{2}} \int_0^\alpha \frac{du}{\sqrt{G(\alpha) - G(u)}} = 1.$$

We define $S(\alpha) \equiv: \int_0^\alpha \frac{du}{\sqrt{G(\alpha)-G(u)}}$. Since G is continuous on $(0, \infty)$, S is continuous on $(0, \infty)$. We will show that

$$\lim_{\alpha \to 0^+} S(\alpha) > \sqrt{2} > \lim_{\alpha \to \infty} S(\alpha).$$

First,

LEMMA 1.
$$S(0^+) = \lim_{\alpha \to 0^+} S(\alpha) \geq \frac{\pi}{\sqrt{2}}.$$

Proof. Let $0 < u < \alpha$. Then

$$\begin{aligned} G(\alpha) - G(u) &= G(\sqrt{\beta}) - G(\sqrt{w}) \\ &= G'(\sqrt{u_*}) \frac{1}{2\sqrt{u_*}} (\beta - w) \text{ for some } w < u_* < \beta \\ &= \frac{1}{2} e^{\sqrt{u_*}} (\alpha^2 - u^2) \leq \frac{1}{2} e^\alpha (\alpha^2 - u^2). \end{aligned}$$

So,

$$S(\alpha)=\int_0^\alpha \frac{du}{\sqrt{G(\alpha)-G(u)}} \geq \int_0^\alpha \frac{du}{\sqrt{\frac{1}{2}e^\alpha(\alpha^2-u^2)}}$$
$$=\sqrt{2}e^{-\frac{\alpha}{2}}\int_0^\alpha \frac{du}{\sqrt{\alpha^2-u^2}}=\frac{\pi}{\sqrt{2e^\alpha}}.$$

Thus we obtain

$$S(0^+)\geq \frac{\pi}{\sqrt{2}}.$$

LEMMA 2. $S'(\alpha)<0$.

Proof. Note that

$$S(\alpha)=\int_0^{\frac{\pi}{2}}(G(\alpha)-G(\alpha\sin\theta))^{-\frac{1}{2}}\alpha\cos\theta\, d\theta.$$

Let $N(\alpha)=G(\alpha)-G(\alpha\sin\theta)$.
Then

$$2S'(\alpha)=\int_0^{\frac{\pi}{2}} N^{-\frac{3}{2}}(2N-\alpha N')\cos\theta\, d\theta.$$

$$\begin{aligned}2N-\alpha N' &= 2(G(\alpha)-G(\alpha\sin\theta))-\alpha(G'(\alpha)-G'(\alpha\sin\theta)\sin\theta)\\ &=(2G(\alpha)-\alpha G'(\alpha))-(2G(\alpha\sin\theta)-\alpha G'(\alpha\sin\theta)\sin\theta).\end{aligned}$$

Let $\gamma(x)=2G(x)-xG'(x)$.
Then

$$2S'(\alpha)=\int_0^{\frac{\pi}{2}} \frac{\gamma(\alpha)-\gamma(u)}{(G(\alpha)-G(u))^{\frac{3}{2}}}du.$$

$\gamma(x)$ is strictly decreasing on $(0,\infty)$. Thus we have $S'(\alpha)<0$. This completes the proof.

$$2S'(\alpha)=\frac{2}{\alpha}S(\alpha)-\int_0^{\frac{\pi}{2}} N^{-\frac{3}{2}}N'\alpha\cos\theta\, d\theta, \text{ and}$$

$$\frac{S'(\alpha)}{S(\alpha)} = \frac{1}{\alpha} - \frac{\int_0^{\frac{\pi}{2}} N^{-\frac{3}{2}} N' \alpha \cos\theta \, d\theta}{2\int_0^{\frac{\pi}{2}} N^{-\frac{3}{2}} N \alpha \cos\theta \, d\theta} \equiv: \frac{1}{\alpha} - c_*.$$

It is obvious that $G < G'$ for all $\alpha > 0$ and $\theta \in (0, \frac{\pi}{2})$.
Noting that $G(\alpha \sin\theta) = G'(\alpha \sin\theta) - (e^{\alpha \sin\theta} - 1)$, we have

$$c_* \geq \frac{1}{2}, \text{ i.e., } \frac{d}{d\alpha}(lnS(\alpha)) = \frac{S'(\alpha)}{S(\alpha)} \leq \frac{1}{\alpha} - \frac{1}{2}.$$

Integrating the inequality above from 1 to α give us

$$S(\alpha) \leq S(1)\frac{\alpha}{e^{\frac{\alpha-1}{2}}}.$$

But the right hand side in the above inequality has limit zero as α tends to infinity. Thus $S(\infty) = \lim_{\alpha\to\infty} S(\alpha) = 0$. So we conclude that there exists unique $\alpha \in (0, \infty)$ such that $S(\alpha) = \sqrt{2}$. This complete the proof of the main Theorem.

It is obvious to prove the next corollary from the equation and Figures 1 and 2.

COROLLARY 3. *There is a unique $h = h_0 \in (0, \infty)$ such that the problem $u'' + ue^u = 0$ $(x \in \mathbb{R})$ has the solution satisfying the following:*

(a) $H(u, u') = G(u) + \frac{1}{2}(u')^2 = h_0$.
(b) $\lim_{|x|\to\infty} u(x) = -\infty$.
(c) $\lim_{|x|\to\infty} u'(x) = 0$.
(d) *For each $h > h_0$, there is a unique $c(h) > 0$ such that $\lim_{x\to\pm\infty} u(x) = \pm c(h)$ respectively.*
(e) *For each $h \in (0, h_0)$, the solution is periodic with period $T(h)$ converging to 2π as $h \to 0$.*

The last statement (e) above Corollary is obvious since $(0, 0)$, the equilibrium point of (5), is center by the linear analysis. In fact, $S(0^+) = \frac{\pi}{\sqrt{2}}$.

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