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ON EINSTEIN WARPED PRODUCTS WITH COMPACT RIEMANNIAN BASE

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1. Introduction

The concept of warped product manifolds was introduced by Bishop and O'Neill in [3], where it served to provide a class of complete Riemannian manifolds with everywhere negative sectional curvature. In [2], they raise a question on Riemannian warped product as follows : Does there exist a compact Einstein warped product with nonconstant warping function? And they prove that it is not possible to find such examples with one or two dimensional Riemannian base. For pseudo-Riemannian warped product, K. Easely proves that if $M = B \times_f F$ is a 4-dimensional Ricci flat warped product with base B a surface of constant curvature, then M is simply a product manifold([1] or [5]).

In this paper, we study Einstein warped product space with compact Riemannian base. As a result, we prove some analogous theorems to the above theorems.

2. Preliminaries

In this section, we recall some properties of warped product manifolds. Complete details may be found in [3], [7].

DEFINITION 2.1. Let $(B, g_B), (F, g_F)$ be a pseudo-Riemannian manifold and let $f > 0$ be a smooth function on B. We denote by π and σ the projections of $B \times F$ onto B and F, respectively. The

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warped product $M = B \times_f F$ is the product manifold $M = B \times F$ furnished with metric tensor g defined by

$$g = \pi^*(g_B) + (f \circ \pi)^2 \sigma^*(g_F),$$

where (*) denotes pullback. The function f will be called the *warping function.*

A warped product $B \times_f F$ with a constant function f can be regarded as a product manifold $B \times \tilde{F}$, where the fibre $\tilde{F}$ is just F with metric $\tilde{g}_F$ given by $\frac{1}{f^2} g_F$. If $f = 1$, then $B \times_f F$ reduces to a product manifold.

B is called the *base* of (M, g) and F the *fiber.* The *lift* of $f \in \mathcal{F}(B)$ to M is $\tilde{f} = f \circ \pi$, defining $\tilde{f} \in \mathcal{F}(M)$. We shall denote the lifted function by f as well, when no ambiguity results. Vectors tangent to leaves are *horizontal*; vectors tangent to fibers are *vertical.* The notion of lift of a vector field on B or F to $B \times F$ may be defined in the usual way, and the set of all such lifts is denoted as usual by $\mathcal{L}(B)$ and $\mathcal{L}(F)$, respectively. Typically we use the same notation for a vector field and for its lift. Let ∇^M denote the Levi-Civita connection on M. We also denote by H^f the lift to M of the Hessian of f.

Now we state the following three lemmas for warped product whose proofs can be found in [7].

LEMMA 2.2. *Let $M = B \times_f F$ be a warped product with Riemannian curvature tensor R^M. If $X, Y, Z \in \mathcal{L}(B)$ and $U, V, W \in \mathcal{L}(F)$, then*

(1) *$R^M_{XY} Z \in \mathcal{L}(B)$ is the lift of $R^B_{XY} Z$ from B.*

(2) *$R^M_{VX} Y = (\frac{H^f(X,Y)}{f}) V$, where H^f is the Hessian of f.*

(3) *$R^M_{XY} V = R^M_{VW} X = 0$.*

(4) *$R^M_{XV} W = (\frac{g(V,W)}{f}) \nabla^M_X (\nabla f)$.*

(5) *$R^M_{VW} U = R^F_{VW} U - (\frac{g(\nabla f, \nabla f)}{f^2}) \{g(V,U)W - g(W,U)V\}$.*

LEMMA 2.3. *On a warped product manifold $M = B \times_f F$ with $d = dim F$, let X, Y be horizontal and V, W vertical. Then*

(1) *$Ric^M(X,Y) = Ric^B(X,Y) - \dfrac{d}{f} H^f(X,Y)$.*

(2) $Ric^M(X,V) = 0$.
(3) $Ric^M(V,W) = Ric^F(V,W) - g(V,W)f^\sharp$, where $f^\sharp = \dfrac{\Delta f}{f} + (d-1)\dfrac{g_B(\nabla f, \nabla f)}{f^2}$, and $\Delta f = C(H^f) = trace(H^f)$ is the Laplacian on B.

On the given warped product manifold $M = B \times_f F$, we also write S^B for the pullback by π of the scalar curvature S^B of B and similarly for S^F.

LEMMA 2.4. *If S^M is the scalar curvature of $M = B \times_f F$ with $d = dimF$, then*
(1) $S^M = S^B + \frac{S^F}{f^2} - 2d\frac{\Delta f}{f} - d(d-1)\frac{g_B(\nabla f, \nabla f)}{f^2}$.
(2) $f^\sharp = \Delta(f^d)/(f^d d)$ *,where Δ is the Laplacian on B.*

3. Main Theorems

First of all, we characterize the warped product with compact Riemannian base which has constant sectional curvature as follows(compare with Theorem 2 in [6]):

THEOREM 3.1. *Let (M,g) be a warped product $B \times_f F$ of r-dimensional $(1 < r < n)$ Riemannian manifold B and $(n-r)$-dimensional manifold F. Suppose that M has constant sectional curvarure K^M. If the base B is compact (in the case, $K^M \leq 0$), or B is complete(in the case, $K^M > 0$) , then $K^M = 0$ and the warped product M simply becomes a Riemannian product.*

Proof. Since M has constant sectional curvature K^M, Lemma 2.2 (1) shows that the base B has constant sectional curvature K^M. Hence we have for $X, Y \in TB$

$$Ric^M(X,Y) = (n-1)K^M g_B(X,Y),$$
$$Ric^B(X,Y) = (r-1)K^M g_B(X,Y).$$

Therefore Lemma 2.3 implies that

$$H^f(X,Y) = -K^M f g_B(X,Y). \tag{3.1}$$

By contracting the both sides of (3.1) we obtain

$$\Delta f = -rK^M f. \tag{3.2}$$

Note that in case, K^M is positive and B is complete, the base B is compact. Since the warping function f is positive, by integrating the both sides of (3.2) on B, we have $K^M = 0$. Hence (3.2) again shows that f is constant. This completes the proof.

If the base B is noncompact, there exist abundant examples of nontrivial warped product which has constant negative sectional curvature. For example, it is well-known that the hyperbolic space form $H^n(-a^2)$ of sectional curvature $-a^2$ can be represented as warped products :

$$R \times_{e^{at}} R^{n-1}, \quad or \quad R \times_{\cosh at} H^{n-1}(-a^2).$$

Hence $H^n(-a^2)$ can be written as follows :

$$H^2(-a^2) \times_f H^{n-2}(-a^2),$$

where the warping function f is given by $coshat_1 coshat_2$ on the hyperbolic plane $H^2(-a^2)=R \times_{\cosh at_1} R$

Recall that a pseudo-Riemannian manifold N is an Einstein manifold provided $Ric = fg$ for some $f \in C^\infty(M)$. If N is connected and $dimN \geq 3$, then f is constant ([2],[7]). In [2], they showed that if M is an Einstein warped product $B \times_f F$ where the base B is compact Riemannian and $\dim B = 1$ or 2, then f is constant, that is, the warped product simply becomes a Riemannian product. In the case of an Einstein warped product over a 2-dimensional basis B, they used the fact that the Hessian tensor H^f of the warping function f is proportional to the metric tensor g_B on the base B([2], pp. 270).

Now we prove an analogous theorem to the above for a warped product with base B of dimension ≥ 3 under the condition that the Hessian of f is proportional to the metric tensor g_B of B.

THEOREM 3.2. *Let* (M, g) *be a connected Einstein warped product* $B \times_f F$ *of* r*-dimensional* $(3 \leq r < n)$ *Riemannian manifold* B *and* $(n-r)$*-dimensional pseudo-Riemannian manifold* F.

If (B, g_B) is compact and the Hessian of f is proportional to the metric tensor g_B, then the warped product M simply becomes a product manifold.

Proof. Since $H^f(X,Y) = (\Delta f/r)g_B(X,Y)$ on B and $Ric^M = \lambda g$, Lemma 2.3 (1) shows that for $X, Y \in TB$, $Ric^B(X,Y) = (\lambda + \frac{d\Delta f}{rf})g_B(X,Y)$. By the hypothesis $\dim B \geq 3$, the second Bianchi identity implies that $\mu = \lambda + \frac{d\Delta f}{rf}$ is constant, that is, the warping function f satisfies the following:

$$\Delta f = \frac{r}{d}(\mu - \lambda)f. \tag{3.3}$$

By integrating the both sides of (3.3) over B, we see that μ is equal to λ. Hence (3.3) again shows that f is constant. This completes the proof.

For pseudo-Riemannian Einstein warped product, K. Easley showed that if $M = B \times_f F$ is a four-dimensional Ricci flat pseudo-Riemannian warped product with base B a surface of constant curvature, then M is simply a product manifold $B \times F$ where B and F are flat two-manifolds([1], p. 127, or [5]).

If the base B of a warped product is a compact Riemannian manifold with constant scalar curvature, then we may prove the following:

THEOREM 3.3. *Let (M, g) be a connected Einstein warped product $B \times_f F$ of connected r-dimensional $(1 < r < n)$ Riemannian manifold B and $(n-r)$-dimensional pseudo-Riemannian manifold F. If B is compact with constant scalar curvature S^B, then the warped product simply becomes a product manifold.*

Proof. Since $Ric^M(X,Y) = \lambda g(X,Y)$ for some constant λ, by contracting the equation (1) of Lemma 2.3 on B, we see that the warping function f satisfies

$$\Delta f = \frac{1}{d}(S^B - \lambda r)f. \tag{3.4}$$

Integrating the both sides of (3.4) on B implies that S^B is equal to λr. Hence (3.4) again shows that the warping function f is constant. This completes the proof.

Let $M = B\times_f F$ be a pseudo-Riemannian warped product with $\dim B \geq 2$ and $\dim F = 1$. Then in [5], K. Easely showed that for M to be Ricci flat, it is necessary and sufficient that

(1) $Ric^B(X,Y) = \frac{1}{f}H^f(X,Y)$ for all $X, Y \in TB$ and

(2) $\Delta f = 0$ on B.

Hence if the base B is compact Riemannian, then (2) shows that f is constant, that is, the warped product is simply a product manifold. In case the fiber is of 2-dimensional we prove an analogous theorem to the above as follows:

THEOREM 3.4. *Let (M,g) be a Ricci flat warped product $B\times_f F$ of r-dimensional connected compact Riemannian manifold B and 2-dimensional pseudo-Riemannian surface F. Then the warped product M is simply a product manifold.*

Proof. From the equation (3) of Lemma 2.3, it follows that for all vector field V, W on F

$$Ric^F(V,W) = f^2 f^\sharp g_F(V,W), \tag{3.5}$$

where $f^\sharp = \dfrac{\Delta f}{f} + (d-1)\dfrac{g_B(\nabla f, \nabla f)}{f^2}$.

Since $Ric^F(V,W)$ and $g_F(V,W)$ are functions on F, from (3.4) we see that $f^2 f^\sharp$ must be a constant λ. Since $\dim F = 2$, the equation (2) of Lemma 2.4 shows that the warping function f satisfies

$$\Delta(f^2) = 2\lambda. \tag{3.6}$$

Integrating the both sides of (3.6) shows that $\lambda = 0$, hence again we see that f is constant. This completes the proof.

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