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MICHAEL'S SELECTION THEORIES AND THEIR APPLICATIONS

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Abstract In this paper, we focus on the convex-valued selection theorem out of four main selection theorems; zero-dimensional, convex-valued, compact-valued, finite-dimensional theorems based on Michael's papers. We prove some theorems about lower semi-continuous set-valued mappings, and derive some applications to closed continuous set-valued mappings and to functional analysis. We also give a partial solution to the open problem posed by Engelking, Heath, and Michael.

1. Introduction

Let X and Y be topological spaces, and 2^Y be the family of nonempty subsets of Y. Also, we let

$$\mathcal{F}(Y) = \{S \in 2^Y : S \text{ is closed}\}.$$

In particular, if Y is a Banach space, then we let

$$\mathcal{F}_c(Y) = \{S \in 2^Y : S \text{ is closed and convex}\},$$

$$\mathcal{C}(Y) = \{S \in \mathcal{F}(Y) : S \text{ is compact or } S = Y\}.$$

A map $\varphi : X \to 2^Y$ is called a *set-valued mapping* (or a *carrior*). A *selection* for $\varphi : X \to 2^Y$ is a continuous map $f : X \to Y$ such that $f(x) \in \varphi(x)$ for every $x \in X$.

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A set-valued mapping $\varphi : X \to 2^Y$ is called *lower semi-continuous* (*upper semi-continuous*) or l.s.c. (u.s.c.) if for every open set (resp. closed) V of Y,

$$\varphi^{-1}\{V\} = \{x \in X : \varphi(x) \cap V \neq \emptyset\}$$

is open (resp. closed) in X.

A set-valued mapping $\varphi : X \to 2^Y$ is called *continuous* if it is both u.s.c. and l.s.c..

A set-valued mapping $\psi : X \to 2^Y$ is a selection for $\varphi : X \to 2^Y$ if $\psi(x) \subset \varphi(x)$ for every $x \in X$.

Most of the classical Michael's selection theorems establish that existence of continuous selections for l.s.c. set-valued mapping $\varphi : X \to \mathcal{F}(Y)$ is equivalent to some higher separation axioms (i.e., paracompactness, collectionwise normality, normality, etc.) of X. One of the theorems is the following.

THEOREM 1.1. *[9, Theorem 3.2″] For a T_1 space X the following are equivalent:*

(a) X is paracompact.

(b) If Y is a Banach space, then every l.s.c. $\varphi : X \to \mathcal{F}_c(Y)$ admits a selection.

This theorem characterizes paracompactness using continuous functions rather than covering properties. Since the continuous functions are relatively easier to handle covering properties, the above Michael's theorem is important and applicable to certain proofs.

Recall that a space X is metacompact(or weakly paracompact) if every open cover of X has a point-finite open refinement.

Every countably compact metacompact space is compact (see [4]).

Choban [2, 3] also characterized weak paracompactness (=metacompactness), which is weaker than paracompactness, using a continuous function.

An extension problem is one of the important branch in general topology. As we will see, selection theories are very closely related to extension problems. For these, we first consider the following general extension problem [9]:

Let X and Y be two topological spaces with $A \subset X$ closed, and let $f : A \to Y$ be continuous. Under what conditions on X and Y, does f have a continuous extension over X (or at least over some open $U \supset A$) ?

Related to this question, we have the following Tietze's extension theorem.

THEOREM 1.2. *Let X be a normal space. Then for every closed $A \subset X$ and every continuous function $f : A \to \mathbb{R}$, there exists a continuous extension of f over X.*

The following theorem is so called a *selection extension property.*

THEOREM 1.3. *[9] Let X and Y be topological spaces and $\mathcal{S} \subset 2^Y$ be a collection containing all single-ton sets. Then the following are equivalent.*

(i) If $\varphi : X \to \mathcal{S}$ is a l.s.c. set-valued mapping, then for every closed $A \subset X$, each selection for $\varphi \mid A$ can be extended to a selection for φ.

(ii) Every l.s.c. set-valued mapping $\varphi : X \to \mathcal{S}$ has a selection.

Theorem 1.3 reduces the problem which deals with extending a selection to the simpler problem of merely finding one.

If $A \subset X$ is closed, then $\varphi : X \to 2^Y$ has the *selection extension property*, or SEP, at A if every selection for $\varphi|A$ extends to a selection for φ; if φ has the SEP at every closed $A \subset X$, then we simlpy say that φ has the SEP.

QUESTION 1.4. *[9] Under what conditions on X, $A \subset X$, and $\varphi : X \to 2^Y$, can every selection for $\varphi|A$ be extended to a selection for φ, or at least for $\varphi|U$ for some open $U \supset A$?*

One of the results related to this question is the following theorem which strengthens some known selection theorems for set-valued mappings $\varphi : X \to 2^Y$ by eliminating all assumptions, except lower semi-continuity, an arbitrary countable subset of X.

THEOREM 1.5. *[12] If X is countable and regular, Y is first-countable, and $\varphi : X \to 2^Y$ is l.s.c., then φ has SEP.*

We introduce properties of l.s.c. set-valued mappings which are used later on in this paper.

PROPOSITION 1.6. *[9] A set-valued mapping $\varphi : X \to 2^Y$ is l.s.c. if and only if for each $x \in X$, $y \in \varphi(x)$, and a neighborhood V of y, there exists a neighborhood U of x such that $\varphi(x') \cap V \neq \emptyset$ for every $x' \in U$.*

PROPOSITION 1.7. *[9] If $\varphi : X \to 2^Y$ is l.s.c.,and if $\psi : X \to 2^Y$ is such that $\psi(x) = \overline{\varphi(x)}$ for every $x \in X$, then ψ is l.s.c..*

QUESTION 1.8. *(Question 3.3 in [13]) Does Theorem 1.1 remain true if $K \subset Y$ is only assumed to be convex G_δ ?*

The answer to this question is "yes" if $dimX$ (Lebesgue covering dimension) is finite (See 3.6 in [13]). Moreover, Gutev [6] showed that if X is a countable-dimensional metric space, E is a Banach space, and $Y \subset E$ is a G_δ-subset, then every l.s.c. $\varphi : X \to \mathcal{F}_c(Y)$ admits a single- valued continuous selection.

2. Properties of Lower Semi-continuous Mappings

The main purpose of this section is to prove a generalized theorem about lower semi-continuous set-valued mappings (Theorem 2.8) and to derive some applications to closed continuous set-valued mappings.

Recall that if X and Y are topological spaces, and if 2^Y denotes the collection of nonempty subsets of Y, then a set-valued mapping $\varphi : X \to 2^Y$ is called l.s.c. if for every open subset V of Y $\varphi^{-1}\{V\} = \{x \in X : \varphi(x) \cap V \neq \emptyset\}$ is an open subset of X.

A (single-valued) mapping $\varphi : X \to Y$ may be considered as a special set-valued mapping of X into 2^Y such that each $\varphi(x)$ is a one-point subset of Y. Then this set-valued mapping is l.s.c. iff the mapping φ is continuous because $\varphi^{-1}\{V\} = \varphi^{-1}(V)$ for every subset V of Y.

We first establish some terminology and notation used throughout this section. We assume that a space is always a T_1 topological space. Each mapping $f : X \to Y$ from a space X into a space Y is a continuous mapping and will always be denoted by lower case letters.

A set-valued map (or a multi-map) $\varphi : X \to 2^Y$ will signify a function from the set X into the set 2^Y of all nonempty subsets of

Y as in Section 1. If A is a subset of X, then $\varphi^*(A) = \bigcup\{\varphi(a) : a \in A\}$.

An inverse set-valued map $\varphi^{-1} : \varphi^*(X) \to X$ can be defined whenever $\varphi^{-1}\{\{y\}\}$ is a closed set for each $y \in \varphi^*(X)$ by setting $\varphi^{-1}(y) = \varphi^{-1}\{\{y\}\}$.

DEFINITION 2.1. [8] A set-valued map $\varphi : X \to 2^Y$ is said to have *compact (closed) inverses* iff for each $y \in \varphi^*(x)$ $\varphi^{-1}\{\{y\}\}$ is compact (closed).

DEFINITION 2.2. [8] A set-valued map $\varphi : X \to 2^Y$ is said to be *closed* iff for each $y \in Y$ and an open subset W in X with $\varphi^{-1}\{\{y\}\} \subset W \subset X$, there is an open set V containing y with $\varphi^{-1}\{\{y\}\} \subset \varphi^{-1}\{V\} \subset W$.

We say that φ is *open* iff for each open set U in X, $\varphi^*(U)$ is open in Y.

THEOREM 2.3. *(Theorem 3.1 in [8]) If $\varphi : X \to 2^Y$ is a set-valued map, then the following are equivalent.*

(1) φ preserves closed sets; i.e., for each closed subset C of X, $\varphi^*(C)$ is closed in Y,

(2) φ is closed, and

(3) for each open set W in X, the set $V = \{y \in Y | \varphi^{-1}\{\{y\}\} \subset W\}$ is open in Y.

Now let us consider the problem concerning metrizability of the image of a metrizable space X under a continuous closed map. In this area, the following Stone-Morita-Hanai Theorem is the most interesting.

THEOREM 2.4. *[4] If $f : X \to Y$ is a continuous closed mapping of a metrizable space X onto a space Y, then Y is metrizable iff the boundary of $f^{-1}(y)$ is compact for each point $y \in Y$.*

The following well-known theorem is due to E. Michael.

THEOREM 2.5. *Let X be paracompact, Y locally compact or first countable, and $f : X \to Y$ continuous and closed. Then $bdy f^{-1}(y)$ (the boundary of $f^{-1}(y)$) is compact for every $y \in Y$.*

THEOREM 2.6. *(Theorem 3.5 in [8]) If (X, d) is a metric space, Y is a first countable space, and $\varphi : X \to 2^Y$ is a l.s.c., closed set-valued map, then, for each y in Y, closure $(bdy\varphi^{-1}\{\{y\}\}) \cap \varphi^{-1}\{\{y\}\}$ is compact in X.*

QUESTION 2.7. *[8] The various theorems make it clear that closed set-valued maps generalize closed single-valued maps. We may ask, in a general way, how much of the theory of closed single-valued maps can be extended to set-valued maps?*

In particular, can we generalize Theorem 2.5 to a set-valued map theorem?

Related to this question, we have a result. A similar strategy for proving Theorem 2.6 works for the following.

THEOREM 2.8. *If X is a collectionwise normal space, Y is a first countable space, and $\varphi : X \to \mathcal{F}(Y)$ is l.s.c., then for each $y \in Y$,*

$$bdy(\varphi^{-1}\{\{y\}\}) \cap \varphi^{-1}\{\{y\}\}$$

is countably compact.

Proof. Let $y \in Y$ and let $\{V_n : n \in \omega\}$ be a base of open neighborhoods of y. Suppose by way of contradiction that $A = bdy(\varphi^{-1}\{\{y\}\}) \cap \varphi^{-1}(y)$ is not countably compact. Then there is a closed discrete set $D \subset bdy(\varphi^{-1}\{\{y\}\})$ of cardinality $\aleph_0$. Enumerate $D = \{x_n : n \in \omega\}$. Since X is collectionwise normal, there is a discrete family $\mathcal{D} = \{U_n : n \in \omega\}$ of open subsets of X such that $x_n \in U_n$ for every $n \in \omega$. By reducing U_n if necessary, we may assume that $\varphi^*(U_n) \subset V_n$ for each $n \in \omega$(since φ is l.s.c.). Also since $x_n \in bdy(\varphi^{-1}\{\{y\}\})$, $U_n \cap (\varphi^{-1}\{V_n\} \backslash \varphi^{-1}\{\{y\}\}) \neq \emptyset$. Choose for each $n \in \omega$ a point $x'_n \in U_n \cap (\varphi^{-1}\{V_n\} \backslash \varphi^{-1}\{\{y\}\})$. Then $\varphi(x'_n) \in V_n$ and consequently y belongs to the closure of the set $Q = \{\varphi(x'_n) : n \in \omega\}$ which is the image of the set $P = \{x'_n : n \in \omega\}$ under φ. But $x'_n \notin \varphi^{-1}\{\{y\}\}$ for each $n \in \omega$ so that $y \notin Q$.

Hence Q is not closed although P is closed(because the family $\{U_n : n \in \omega\}$ is discrete and $x'_n \in U_n$). Since $\varphi^*(P) = Q$, this contradicts the fact (Theorem 2.3) that φ is closed.

In the rest of this section, we prove some simple results about lower semi-continuous set-valued mappings.

LEMMA 2.9. *[15] Let X, Y be topological spaces, $\varphi : X \to 2^Y$ a l.s.c. set-valued mapping with dense range. If D is dense in X, then $\varphi(D)$ is dense in Y.*

THEOREM 2.10. *Let X be a separable space and Y be a topological spaces. Let $\varphi : X \to 2^Y$ be l.s.c. with dense range. Then Y is also separable.*

Proof. Since X is separable, there exists a countable dense subset D of X. Then by Lemma 2.9, $\varphi(D)$ is dense in Y, and clearly $\varphi(D)$ is countable. Hence Y is separable.

If $\varphi : X \to 2^Y$, then we say that $\phi \subset \varphi$ if $\phi : X \to 2^Y$ and $\phi(x) \subset \varphi(x)$ for every $x \in X$.

THEOREM 2.11. *Suppose $\varphi : X \to 2^Y$ is l.s.c. and $\phi : X \to 2^Y$ with $\phi \subset \varphi$ and $\overline{\phi(x)} = \varphi(x)$ for every $x \in X$. Then ϕ is also l.s.c..*

Proof. Since φ is l.s.c., so is $\overline{\varphi}$ ([Proposition 1.7]). But since $\varphi(x) = \overline{\phi}(x)$ for every $x \in X$, $\overline{\varphi} = \overline{\overline{\phi}} = \overline{\phi}$ and so $\overline{\phi}$ is l.s.c.. Therefore ϕ is l.s.c. by Proposition 1.7 again.

DEFINITION 2.12. A set-valued map $\varphi : X \to 2^Y$ is said to be *perfect* if φ is continuous closed, and has compact inverses, i.e., for each $y \in \varphi^*(X)$, $\varphi^{-1}(y)$ is compact.

THEOREM 2.13. *[8] Suppose that $\varphi : X \to \mathcal{F}(Y)$ is a continuous closed set-valued map. Then φ has compact inverses if and only if for each compact set $K \subset Y$, $\varphi^{-1}\{K\}$ is compact in X.*

THEOREM 2.14. *If $\varphi : X \to 2^Y$ is perfect, then for every compact set $K \subset Y$, $\varphi^{-1}\{K\}$ is compact in X.*

Proof. It follows directly from Definition 2.12 and Theorem 2.13.

By a straightforward modification of Theorem 3.3 in [8], we obtain the following.

THEOREM 2.15. *If $\varphi : X \to 2^Y$ is a closed continuous set-valued map and for each $y \in \varphi^*(X)$, $\varphi^{-1}(y)$ is countably compact, then for every countably compact set $K \subset Y$, $\varphi^{-1}\{K\}$ is countably compact.*

3. Applications of Michael's Selection Theorems

We shall now present important applications of the Michael selection theorem. In this section, we also give a partial solution (Theorem 3.8) to the open problem posed by Engelking, Heath, and Michael [5]. To begin with, let us introduce the open mapping theorem in functional analysis.

THEOREM 3.1. *If Y and X are Banach spaces, and if T is a continuous linear transformation from Y onto X, then T is open.*

The open mapping theorem and Michael selection theorem now imply the following three corollaries.

COROLLARY 3.2. *(Bartle-Graves) If Y and X are Banach spaces, and if T is a continuous linear transformation from Y onto X, then there exists a continuous map $f : X \to Y$ such that $f(x) \in T^{-1}(x)$ for every $x \in X$.*

Proof. See [9].

It is well-known from functional analysis that if T is a linear transformation from finite dimensional vector space X (over $\mathbb{R}$ or $\mathbb{C}$) into a vector space Y, then X is isomorphic to the direct sum of Y with the kernel of T, i.e. $Y \bigoplus Ker(T)$. But if both X and Y are infinite dimensional, then this kind of isomorphism no longer exists. Although, it turns out that there exists a homeomorphism between X and $Y \bigoplus Ker(T)$.

COROLLARY 3.3. *[1] If T is a continuous linear transformation that maps a Banach space X onto a Banach space Y, then the space X is homeomorphic to the direct sum $Y \bigoplus Ker(T)$.*

More generally, we have the following corollary.

COROLLARY 3.4. *If Y and X are Banach spaces, and if T is a continuous linear transformation from Y onto X, then there exists a continuous map $f : X \to Y$ such that $T \circ f = 1_X$ and the map $h : Y \to (KerT) \times X$ defined by $h(y) = (y - f(T(y)), T(y))$ is a homeomorphism.*

Proof. Define $\varphi : X \to 2^Y$ by $\varphi(x) = T^{-1}(x)$. Then for each open $V \subset Y$, $\varphi^{-1}\{V\} = T(V)$. So by Theorem 3.1, $\varphi^{-1}\{V\}$ is open in X. Therefore φ is l.s.c. by Example 1.1* in [9]. Since T is onto, each $T^{-1}(x)$ is non-empty. Also, since each $T^{-1}(x)$ is convex and closed by linearity and continuity of T respectively, φ is a l.s.c. set-valued mapping of X into $\mathcal{F}_c(Y)$. Therefore by Theorem 1.1, there exists a continuous map $f : X \to Y$ such that $f(x) \in \varphi(x) = T^{-1}(x)$. The rest of the proof easily follows.

We now introduce a characterization of weak paracompactness using a continuous function.

THEOREM 3.5. *[2] For a T_1-space X, the following are equivalent.*

(a) X is weakly paracompact(=metacompact),

(b) If Y is a complete metric space, then every l.s.c. set-valued map $\varphi : X \to \mathcal{F}_c(Y)$ has a compact-valued lower semi-continuous selection.

REMARK 3.6. Using the above Theorem 3.5, we can reprove Theorem 1.1. We include its reproof below because of different nature of its proof.

Since X is paracompact, it is weakly paracompact and collectionwise normal [4]. Note that every Banach space is a complete metric space. Thus by Theorem 3.5, φ has a l.s.c. $\phi : X \to \mathcal{C}(Y)$ such that $\phi(x) \subset \varphi(x)$ for every $x \in X$. So, by Theorem 3.2′ in [9], ϕ has a single-valued selection and this selection is also a selection for φ.

Engelking, Heath, and Michael [5] proved that every zero-dimensional, complete metric space admits a continuous selection. In [5], they pointed out the completeness cannot be dropped in the theorem above. In fact, they proved that the space of rational numbers $\mathbb{Q}$ does not have any continuous selection ([5], Theorem 6.1). In connection with these results, they asked whether a metrizable space X which admits a continuous selection must be completely metrizable. The question is still unknown.

Hattori and Nogura ([7], Theorem 2.3) showed that there exists no continuous selection on the space of all nonempty clopen subsets of $\mathbb{Q}$. It is well-known that every completely metrizable space is Baire [4]. It is also known that a metrizable space X is hereditarily Baire if and only if X contains no closed subspace which is homeomorphic to $\mathbb{Q}$. Hence, it follows from [4, Theorem 6.1] that every metrizable space X admitting a continuous selection is hereditarily Baire.

THEOREM 3.7. *([7], Theorem 1.2) If a regular space X admits a continuous selection, then X is a hereditarily Baire space.*

Related to the problem mentioned above, we have a partial answer in the following.

THEOREM 3.8. *If X is countable and regular, Y is first-countable and $\varphi : X \to \mathcal{S}$ is l.s.c., where $\mathcal{S} \subset 2^Y$ contains all one point subset of elements of Y, then X is hereditarily Baire.*

Proof. Under the assumption, φ has SEP by Theorem 1.5 and φ admits a selection by Theorem 1.3. Hence X is a hereditarily Baire space by Theorem 3.7.

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