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PRO–TORSION PRODUCTS AND ČECH HOMOLOGY GROUPS

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Abstract We find some properties of the pro-torsion products. Under the suitable conditions, we also show that the map $\bar{H}_p(\mathcal{X};G) \to \bar{H}_p^{s(r)}(\mathcal{X};G)$ is an isomorphism and the n-th homotopy group of X is isomorphic to the n-th Čech homology group.

1. Introduction

The nice results of the strong homology and derived functor on the inverse systems were introduced by J. T. Lisica, S. Mardešić [4,5] and G. Nöbeling [8]. In particular, C. A. McGibbon and R. Steiner [7] studied the first derived functor of the inverse limit and phantom maps which is an important branch in algebraic topology. N. Shekutkovski [9] showed that the coherent category $CPH - Top$ is a full subcategory of Coh. The authors [2,3] find an isomorpism which is useful as the computation of a derived group by deleting the suitable terms and construct a special exact sequence.

In this paper, we find some properties (Theorem 2.5, Theorem 2.8) of the pro-torsion products of the given inverse systems. We also show that the map $\bar{H}_p(\mathcal{X};G) \to \bar{H}_p^{s(r)}(\mathcal{X};G)$ is an isomorphism (Theorem 3.2). Moreover, we prove (Theorem 3.3) that if $\pi_{n+1}(\mathcal{X},*)$ has the Mittag-Leffler property and $(\mathcal{X},*)$ is $(n-1)$-connected, then there is an isomorphism between the n-th homotopy group and the n-dimensional Čech homology group.

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2. Pro-torsion Products of Pro-groups

Let $\mathcal{C}$ be an arbitrary category and let $\mathcal{D} = (D_\lambda, d_{\lambda\lambda'}, \Lambda)$ and $\mathcal{E} = (E_\gamma, e_{\gamma\gamma'}, \Gamma)$ be inverse systems in the given category $\mathcal{C}$. A morphism of inverse systems $f : \mathcal{D} \to \mathcal{E}$ consists of an increasing function $\phi : \Gamma \to \Lambda$ and of morphisms $f_\gamma : D_{\phi(\gamma)} \to E_\gamma, \gamma \in \Gamma$ such that the following diagram commutes

$$\begin{array}{ccc} D_{\phi(\gamma)} & \xleftarrow{d_{\phi(\gamma)\phi(\gamma')}} & D_{\phi(\gamma')} \\ f_\gamma \downarrow & & f_{\gamma'} \downarrow \\ E_\gamma & \xleftarrow{e_{\gamma\gamma'}} & E_{\gamma'} \end{array}$$

for each $\gamma \leq \gamma'$ in Γ. The category $inv - \mathcal{C}$ consists of inverse systems and morphisms of inverse systems. Two morphisms of inverse systems $f, f' : \mathcal{D} \to \mathcal{E}$, given by (ϕ, f_γ) and (ϕ', f'_γ) respectively, are said to be *equivalent* if there exists an increasing function $\varphi : \Gamma \to \Lambda$ such that $\varphi \geq \phi, \phi'$ and

$$f_\gamma \circ d_{\phi(\gamma)\varphi(\gamma)} = f'_\gamma \circ d_{\phi'(\gamma)\varphi(\gamma)}.$$

The category $pro - \mathcal{C}$ consists of inverse systems and equivalence classes of morphisms. We specialize the category $pro - \mathcal{C}$ to the case $\mathcal{C} = Gr$ or Top, where Gr and Top are categories of groups and topological spaces respectively. The object of $pro - Gr$ is said to be *pro-group*, *i.e.* inverse system of groups and group homomorphisms as bonding morphisms.

Let $\mathcal{G} = (G_\lambda, g_{\lambda\lambda'}, \Lambda)$ and $\mathcal{H} = (H_\lambda, h_{\lambda\lambda'}, \Lambda)$ be objects of $pro - Gr$. Then the morphism $f : \mathcal{G} \to \mathcal{H}$ consists of an identity map $id : \Lambda \to \Lambda$ and of a homomorphism $f_\lambda : G_\lambda \to H_\lambda$, $\lambda \in \Lambda$ such that

$$h_{\lambda\lambda'} \circ f_{\lambda'} = f_\lambda \circ g_{\lambda\lambda'}, \ \lambda \leq \lambda'.$$

Let

$$\mathcal{P} : \cdots \to \mathcal{P}^n \xrightarrow{d_{n-1}} \mathcal{P}^{n-1} \to \cdots$$
$$\to \mathcal{P}^1 \xrightarrow{d_0} \mathcal{P}^0 \xrightarrow{e} \mathcal{G} \to 0$$

be a projective resolution of a pro-group $\mathcal{G}$. The *homological dimension* $hd(\mathcal{G})$ [5] of a pro-group $\mathcal{G} \in \text{obj}(pro - Gr)$ is defined as the smallest n such that $\mathcal{G}$ admits a projective resolution of length n,

$$\mathcal{P} : 0 \to \mathcal{P}^n \xrightarrow{d_{n-1}} \mathcal{P}^{n-1} \to \cdots$$
$$\to \mathcal{P}^1 \xrightarrow{d_0} \mathcal{P}^0 \xrightarrow{e} \mathcal{G} \to 0,$$

where $\mathcal{P}^n = (P^n_\lambda, i_{\lambda\lambda'}, \Lambda)$ is an abelian pro-group over Λ.

We consider the tensor product $\mathcal{P}^n \otimes \mathcal{H}$ of the pro-groups $\mathcal{P}^n$ and $\mathcal{H}$.

DEFINITION 2.1. The *pro-tensor product* $\mathcal{P}^n \otimes \mathcal{H}$ of the pro-groups $\mathcal{P}^n$ and $\mathcal{H}$ is defined by

$$\mathcal{P}^n \otimes \mathcal{H} = (\mathcal{P}^n_\lambda \otimes H_\lambda, i_{\lambda\lambda'} \otimes h_{\lambda\lambda'}, \Lambda).$$

Note that $\mathcal{P}^n \otimes \mathcal{H}$ is a pro-group. If we consider the tensor product $\mathcal{P} \otimes \mathcal{H}$ of the projective resolution $\mathcal{P}$ and pro-group $\mathcal{H}$ as the following;

$$\mathcal{P} \otimes \mathcal{H} : \cdots \to \mathcal{P}^{n+1} \otimes \mathcal{H} \xrightarrow{d^*_n} \mathcal{P}^n \otimes \mathcal{H} \to \cdots$$
$$\to \mathcal{P}^1 \otimes \mathcal{H} \xrightarrow{d^*_0} \mathcal{P}^0 \otimes \mathcal{H} \xrightarrow{e_*} \mathcal{G} \otimes \mathcal{H} \to 0,$$

where $d^*_n = d_n \otimes id_{\mathcal{H}}$ and $e_* = e \otimes id_{\mathcal{H}}$, then we have

$$\begin{aligned} d^* \circ d^* &= (d \otimes id_{\mathcal{H}}) \circ (d \otimes id_{\mathcal{H}}) \\ &= (d \circ d) \otimes (id_{\mathcal{H}}) \\ &= 0 \otimes id_{\mathcal{H}} \\ &= 0. \end{aligned}$$

Therefore, $(\mathcal{P} \otimes \mathcal{H}, d^*)$ is a chain complex.

DEFINITION 2.2. The *n-th pro-torsion product* $\bar{T}or_n(\mathcal{G}, \mathcal{H})$ of the given pro-groups $\mathcal{G}$ and $\mathcal{H}$ is defined by

$$\bar{T}or_n(\mathcal{G}, \mathcal{H}) = \ker(d^*_{n-1})/\text{im}(d^*_n).$$

PROPOSITION 2.3. *If $hd(\mathcal{G}) \leq n$, then we have*

$$\bar{T}or_m(\mathcal{G},\mathcal{H}) = \begin{cases} \ker(d^*_{n-1}) & \text{for } m = n \\ 0 & \text{for } m > n \end{cases}$$

Proof. From the definition of pro-torsion product and homological dimension, we complete the proof.

DEFINITION 2.4. A morphism $f : \mathcal{G} \to \mathcal{H}$ is called a *pro-monomorphism* over Λ if $f_\lambda : G_\lambda \to H_\lambda$ is injective for each $\lambda \in \Lambda$. Similarly, a morphism $f : \mathcal{G} \to \mathcal{H}$ is called a *pro-epimorphism* over Λ if $f_\lambda : G_\lambda \to H_\lambda$ is surjective for each $\lambda \in \Lambda$. A morphism $f : \mathcal{G} \to \mathcal{H}$ is called a *pro-isomorphism* over Λ if $f_\lambda : G_\lambda \to H_\lambda$ is bijective for each $\lambda \in \Lambda$. In this case $\mathcal{G}$ and $\mathcal{H}$ are said to be *pro-isomorphic* (written $\mathcal{G} \approx \mathcal{H}$). A pro-group $\mathcal{T}$ is said to be *pro-projective group* if $\mathcal{T}_\lambda$ is projective for each λ.

THEOREM 2.5. *Let*

$$0 \to \mathcal{S} \xrightarrow{s} \mathcal{T} \xrightarrow{t} \mathcal{U} \to 0$$

be an exact sequence of pro-groups and $\mathcal{T}$ be a pro-projective group. Then we have

$$\bar{T}or_n(\mathcal{U},\mathcal{H}) = \bar{T}or_{n-1}(\mathcal{S},\mathcal{H}).$$

for $n \geq 2$.

Proof. Let

$$\mathcal{A} : \cdots \to \mathcal{A}_n \xrightarrow{a_{n-1}} \mathcal{A}_{n-1} \to \cdots$$
$$\to \mathcal{A}_1 \xrightarrow{a_0} \mathcal{A}_0 \xrightarrow{b_0} \mathcal{S} \to 0$$

be a projective resolution of $\mathcal{S}$. Since $b_0 : \mathcal{A}_0 \to \mathcal{S}$ is a pro-epimorphism and $s : \mathcal{S} \to \mathcal{T}$ is a pro-monomorphism, we have the commutative diagram

$$\begin{array}{ccccccc} & & & & 0 & & \\ & & & & \downarrow & & \\ \cdots & \longrightarrow & \mathcal{A}_0 & \xrightarrow{b_0} & \mathcal{S} & \longrightarrow & 0 \\ & & {\scriptstyle s\circ b_0}\downarrow & & \downarrow{\scriptstyle s} & & \\ & & \mathcal{T} & = & \mathcal{T} & & \end{array}$$

and

$$\ker(s \circ b_0) \approx \ker(b_0)$$
$$\mathrm{im}(s \circ b_0) \approx \mathrm{im}(s).$$

Define a pro-group $\bar{\mathcal{A}}_n$ by

$$\bar{\mathcal{A}}_n = \begin{cases} 0 & \text{for } n \leq -2 \\ \mathcal{U} & \text{for } n = -1 \\ \mathcal{T} & \text{for } n = 0 \\ \mathcal{A}_{n-1} & \text{for } n \geq 1 \end{cases}$$

and a morphism $\bar{a}_n, n \in \mathbb{Z}$ (the set of all integers) by

$$\bar{a}_n = \begin{cases} 0 & \text{for } n \leq -2 \\ t & \text{for } n = -1 \\ s \circ b_0 & \text{for } n = 0 \\ a_{n-1} & \text{for } n \geq 1. \end{cases}$$

Then we have the following projective resolution $\bar{\mathcal{A}}$ of $\mathcal{U}$

$$\bar{\mathcal{A}} : \cdots \to \bar{\mathcal{A}}_{n+1} = \mathcal{A}_n \xrightarrow[(=a_{n-1})]{\bar{a}_n} \bar{\mathcal{A}}_n = \mathcal{A}_{n-1} \to \cdots$$
$$\to \bar{\mathcal{A}}_2 = \mathcal{A}_1 \xrightarrow[(=a_0)]{\bar{a}_1} \bar{\mathcal{A}}_1 = \mathcal{A}_0 \xrightarrow[(=sb_0)]{\bar{a}_0} \bar{\mathcal{A}}_0 = \mathcal{T} \xrightarrow{t} \mathcal{U} \to 0.$$

Therefore, we have

$$\begin{aligned} \bar{\mathcal{T}}or_n(\mathcal{U}, \mathcal{H}) &= \ker(\bar{a}_{n-1} \otimes id_{\mathcal{H}})/\mathrm{im}(\bar{a}_n \otimes id_{\mathcal{H}}) \\ &= \ker(a_{n-2} \otimes id_{\mathcal{H}})/\mathrm{im}(a_{n-1} \otimes id_{\mathcal{H}}) \\ &= \bar{\mathcal{T}}or_{n-1}(\mathcal{S}, \mathcal{H}). \end{aligned}$$

for any integer $n \geq 2$.

LEMMA 2.6. *Let $\tilde{\mathcal{A}} = (\tilde{A}_\lambda, \tilde{a}_{\lambda\lambda'}, \Lambda)$, $\tilde{\mathcal{B}} = (\tilde{B}_\lambda, \tilde{b}_{\lambda\lambda'}, \Lambda)$ and $\tilde{\mathcal{C}} = (\tilde{C}_\lambda, \tilde{c}_{\lambda\lambda'}, \Lambda)$ be pro-groups and*

$$\tilde{A}_\lambda \xrightarrow{h_\lambda} \tilde{B}_\lambda \xrightarrow{k_\lambda} \tilde{C}_\lambda \to 0$$

be an exact sequence for each $\lambda \in \Lambda$. Then the induced morphism

$$k \otimes id_{\mathcal{H}} : \tilde{\mathcal{B}} \otimes \mathcal{H} \to \tilde{\mathcal{C}} \otimes \mathcal{H}$$

is a pro-epimorphism, where $k = (k_\lambda) : \tilde{\mathcal{B}} \to \tilde{\mathcal{C}}$.

Proof. Since

$$\tilde{A}_\lambda \xrightarrow{h_\lambda} \tilde{B}_\lambda \xrightarrow{k_\lambda} \tilde{C}_\lambda \to 0$$

is exact, the sequence

$$\tilde{A}_\lambda \otimes H_\lambda \xrightarrow{h_\lambda \otimes id_{H_\lambda}} \tilde{B}_\lambda \otimes H_\lambda \xrightarrow{k_\lambda \otimes id_{H_\lambda}} \tilde{C}_\lambda \otimes H_\lambda \to 0$$

of tensor products is also exact [1] for each $\lambda \in \Lambda$. From the following commutative diagram;

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
\downarrow & & \downarrow & & \downarrow & & \\
\tilde{A}_{\lambda'} \otimes H_{\lambda'} & \xrightarrow{h_{\lambda'} \otimes id_{H_{\lambda'}}} & \tilde{B}_{\lambda'} \otimes H_{\lambda'} & \xrightarrow{k_{\lambda'} \otimes id_{H_{\lambda'}}} & \tilde{C}_{\lambda'} \otimes H_{\lambda'} & \longrightarrow & 0 \\
{\scriptstyle \tilde{a}_{\lambda\lambda'} \otimes h_{\lambda\lambda'}} \downarrow & & {\scriptstyle \tilde{b}_{\lambda\lambda'} \otimes h_{\lambda\lambda'}} \downarrow & & {\scriptstyle \tilde{c}_{\lambda\lambda'} \otimes h_{\lambda\lambda'}} \downarrow & & \\
\tilde{A}_{\lambda} \otimes H_{\lambda} & \xrightarrow{h_{\lambda} \otimes id_{H_{\lambda}}} & \tilde{B}_{\lambda} \otimes H_{\lambda} & \xrightarrow{k_{\lambda} \otimes id_{H_{\lambda}}} & \tilde{C}_{\lambda} \otimes H_{\lambda} & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
\vdots & & \vdots & & \vdots & &
\end{array}$$

we have an exact sequence

$$\tilde{\mathcal{A}} \otimes \mathcal{H} \xrightarrow{h \otimes id_{\mathcal{H}}} \tilde{\mathcal{B}} \otimes \mathcal{H} \xrightarrow{k \otimes id_{\mathcal{H}}} \tilde{\mathcal{C}} \otimes \mathcal{H} \to 0$$

of their pro-tensor products, where $h = (h_\lambda) : \tilde{\mathcal{A}} \to \tilde{\mathcal{B}}$. Therefore, $k \otimes id_{\mathcal{H}}$ is a pro-epimorphism.

LEMMA 2.7. *Let*

$$\begin{array}{ccc} \mathcal{K} & \xrightarrow{l} & \mathcal{L} \\ {\scriptstyle m}\downarrow & & {\scriptstyle n}\downarrow \\ \mathcal{M} & = & \mathcal{M} \end{array}$$

be a commutative diagram of pro-groups in Λ *and* l *be a pro-epimorphism. Then the morphism*

$$\ker(m \otimes id_{\mathcal{H}}) \to \ker(n \otimes id_{\mathcal{H}})$$

is a pro-epimorphism and

$$\ker(m \otimes id_{\mathcal{H}})/\ker(l \otimes id_{\mathcal{H}}) \approx \ker(n \otimes id_{\mathcal{H}}).$$

Proof. Consider the following commutative diagram;

$$\begin{array}{ccc} \mathcal{K} \otimes \mathcal{H} & \xrightarrow{l \otimes id_{\mathcal{H}}} & \mathcal{L} \otimes \mathcal{H} \\ {\scriptstyle m \otimes id_{\mathcal{H}}}\downarrow & & {\scriptstyle n \otimes id_{\mathcal{H}}}\downarrow \\ \mathcal{M} \otimes \mathcal{H} & = & \mathcal{M} \otimes \mathcal{H}. \end{array}$$

By Lemma 2.6, $l \otimes id_{\mathcal{H}}$ is a pro-epimorphism. If we consider the commutative diagram;

$$\begin{array}{ccc} K_\lambda \otimes H_\lambda & \xrightarrow{l_\lambda \otimes id_{H_\lambda}} & L_\lambda \otimes H_\lambda \\ {\scriptstyle m_\lambda \otimes id_{H_\lambda}}\downarrow & & {\scriptstyle n_\lambda \otimes id_{H_\lambda}}\downarrow \\ M_\lambda \otimes H_\lambda & = & M_\lambda \otimes H_\lambda \end{array}$$

for each $\lambda \in \Lambda$, then we have an epimorphism

$$\ker(m_\lambda \otimes id_{H_\lambda}) \to \ker(n_\lambda \otimes id_{H_\lambda})$$

and

$$\ker(m_\lambda \otimes id_{H_\lambda})/\ker(l_\lambda \otimes id_{H_\lambda}) \cong \ker(n_\lambda \otimes id_{H_\lambda})$$

for each $\lambda \in \Lambda$. Thus we also have that

$$\ker(m \otimes id_{\mathcal{H}}) \to \ker(n \otimes id_{\mathcal{H}})$$

is a pro-epimorphism over Λ and

$$\ker(m \otimes id_{\mathcal{H}})/\ker(l \otimes id_{\mathcal{H}}) \approx \ker(n \otimes id_{\mathcal{H}}).$$

THEOREM 2.8. *Under the assumption of Theorem 2.5, we have*

$$\bar{T}or_1(\mathcal{U},\mathcal{H}) \approx ker(s \otimes id_{\mathcal{H}}).$$

Proof. By Lemma 2.6, we have an exact sequence

$$\mathcal{A}_1 \otimes \mathcal{H} \xrightarrow{a_0 \otimes id_{\mathcal{H}}} \mathcal{A}_0 \otimes \mathcal{H} \xrightarrow{b_0 \otimes id_{\mathcal{H}}} \mathcal{S} \otimes \mathcal{H} \to 0,$$

where $\mathcal{A}_0$ and $\mathcal{A}_1$ are pro-projective groups in Λ. Consider the commutative diagram;

$$\begin{array}{ccc} \mathcal{A}_0 \otimes \mathcal{H} & \xrightarrow{b_0 \otimes id_{\mathcal{H}}} & \mathcal{S} \otimes \mathcal{H} \\ {\scriptstyle \bar{a}_0 \otimes id_{\mathcal{H}}} \downarrow & & \downarrow {\scriptstyle s \otimes id_{\mathcal{H}}} \\ \mathcal{T} \otimes \mathcal{H} & = & \mathcal{T} \otimes \mathcal{H}. \end{array}$$

By Lemma 2.7, we obtain

$$\begin{aligned} \bar{T}or_1(\mathcal{U},\mathcal{H}) &= \ker(\bar{a}_0 \otimes id_{\mathcal{H}}/\mathrm{im}(\bar{a}_1 \otimes id_{\mathcal{H}}) \\ &= \ker(\bar{a}_0 \otimes id_{\mathcal{H}})/\mathrm{im}(a_0 \otimes id_{\mathcal{H}}) \\ &= \ker(\bar{a}_0 \otimes id_{\mathcal{H}})/\ker(b_0 \otimes id_{\mathcal{H}}) \\ &\approx \ker(s \otimes id_{\mathcal{H}}). \end{aligned}$$

3. r-stage strong homology groups and Čech homology groups

Let $\mathcal{X} = (X_\lambda, f_{\lambda\lambda'}, \Lambda)$ be an inverse system of topological spaces X_λ and continuous maps $f_{\lambda\lambda'} : X_{\lambda'} \to X_\lambda$, $\lambda \le \lambda'$, over the directed set Λ. Let $\Lambda^n, n \ge 0$, be the set of all $\bar{\lambda} = (\lambda_0, \lambda_1, \cdots, \lambda_n)$, $\lambda_0 \le \lambda_1 \le \cdots \le \lambda_n, \lambda_i \in \Lambda$ and G be an abelian group. Let $\bar{\lambda}_j \in \Lambda^{n-1}$, $0 \le j \le n$ be obtained from $\bar{\lambda} = (\lambda_0, \lambda_1, \cdots, \lambda_n) \in \Lambda^n$ by deleting the j-th factor λ_j. The *strong p-chain group* of $\mathcal{X}$ with coefficients in G is given by

$$\bar{C}_p(\mathcal{X}; G) = \prod_{n=0}^{\infty} \prod_{\bar{\lambda} \in \Lambda^n} C_{p+n}(X_{\bar{\lambda}}; G),$$

where $C_{p+n}(X_{\bar{\lambda}};G)$ is a $(p+n)$-chain group of $X_{\bar{\lambda}} = X_{\lambda_0}$.

Let $pr_{\bar{\lambda}} : \bar{C}_p(\mathcal{X};G) \to C_{p+n}(X_{\bar{\lambda}};G)$ be a projection. If x is an element of $\bar{C}_p(\mathcal{X};G)$, then we denote the element $x_{\bar{\lambda}}$ of $C_{p+n}(X_{\bar{\lambda}};G)$ by

$$x_{\bar{\lambda}} = pr_{\bar{\lambda}}(x).$$

A boundary operator $d_p : \bar{C}_{p+1}(\mathcal{X};G) \to \bar{C}_p(\mathcal{X};G)$ is defined by

$$\begin{cases} (d_p(x))_{(\lambda_0)} = \partial(x_{(\lambda_0)}) \text{ for } n = 0 \\ (-1)^n(d_p x)_{\bar{\lambda}} = \partial(x_{\bar{\lambda}}) - f_{\lambda_0\lambda_1}(x_{\bar{\lambda}_0}) - \sum_{j=1}^{n}(-1)^j x_{\bar{\lambda}_j} \text{ for } n \geq 1, \end{cases}$$

where $x \in \bar{C}_{p+1}(\mathcal{X};G)$. Then we have that $(\bar{C}_*(\mathcal{X};G), d)$ is a chain complex. The *p-dimensional strong homology group* $\bar{H}_p(\mathcal{X};G)$ of the inverse system $\mathcal{X}$ with coefficients in G is defined by

$$\bar{H}_p(\mathcal{X};G) = \ker(d_{p-1})/\mathrm{im}(d_p).$$

For a given chain complex $(\bar{C}_*(\mathcal{X};G), d)$, we can make a subchain complex $(C_*^{(r)}(\mathcal{X};G), d^{(r)})$, $r \geq 0$ of $(\bar{C}_*(\mathcal{X};G), d)$ whose p-th chain group is defined by

$$\bar{C}_p^{(r)}(\mathcal{X};G) = \prod_{n=0}^{r} \prod_{\bar{\lambda}\in\Lambda^n} C_{p+n}(X_{\bar{\lambda}};G),$$

and the boundary operator $d_p^{(r)} : \bar{C}_{p+1}^{(r)}(\mathcal{X};G) \to \bar{C}_p^{(r-1)}(\mathcal{X};G)$ is obtained by the restriction map $d_p|_{\bar{C}_{p+1}^{(r)}(\mathcal{X};G)}$. We have the p-dimensional strong homology group, denoted by $\bar{H}_p^{(r)}(\mathcal{X};G)$, of the subchain complex $(C_*^{(r)}(\mathcal{X};G), d^{(r)})$ and we also have the homomorphisms

$$j_{p*}^{r,r+1} : \bar{H}_p^{(r+1)}(\mathcal{X};G) \to \bar{H}_p^{(r)}(\mathcal{X};G)$$

and

$$j_{p*}^{r} : \bar{H}_p(\mathcal{X};G) \to \bar{H}_p^{(r)}(\mathcal{X};G)$$

induced by the natural projections

$$j_{p\sharp}^{r,r+1} : \bar{C}_p^{(r+1)}(\mathcal{X};G) \to \bar{C}_p^{(r)}(\mathcal{X};G)$$

and

$$j_{p\sharp}^{r} : \bar{C}_p(\mathcal{X};G) \to \bar{C}_p^{(r)}(\mathcal{X};G)$$

respectively. The *r-stage p-dimensional strong homology group* is defined by

$$\bar{H}_p^{s(r)}(\mathcal{X};G) = j_{p*}^{r,r+1}(\bar{H}_p^{(r+1)}(\mathcal{X};G)).$$

In this case, we have an inverse system

$$(\bar{H}_p^{s(r)}(\mathcal{X};G), \bar{j}) = (\bar{H}_p^{s(r)}(\mathcal{X};G), \bar{j}_{p*}^{r,r+1}, \mathbb{N}'),$$

where $\bar{j}_{p*}^{r,r+1} : \bar{H}_p^{s(r+1)}(\mathcal{X};G) \to \bar{H}_p^{s(r)}(\mathcal{X};G)$ is a homomorphism and $\mathbb{N}'$ is the set of all non-negative integers.

Let $H_n(\mathcal{X};G)$ denote the induced inverse system of homology groups $(H_n(X_\lambda;G), \tilde{f}_{\lambda\lambda'}, \Lambda)$. The *n-dimensional Čech homology group* $\check{H}_n(\mathcal{X};G), n = 0, 1, 2, \cdots$ is defined by

$$\check{H}_n(\mathcal{X};G) = \varprojlim_{\lambda}(H_n(\mathcal{X};G)).$$

An inverse system $\mathcal{X} = (X_\lambda, f_{\lambda\lambda'}, \Lambda)$ is said to be *Mittag-Leffler property* if every $\lambda \in \Lambda$ admits a $\lambda' \geq \lambda$ such that

$$f_{\lambda\lambda''}(X_{\lambda''}) = f_{\lambda\lambda'}(X_{\lambda'})$$

for any $\lambda'' \geq \lambda'$.

Let $\mathcal{G} = (G_\lambda, g_{\lambda\lambda'}, \Lambda)$ be an inverse system of abelian groups G_λ and group homomorphisms $g_{\lambda\lambda'} : G_{\lambda'} \to G_\lambda$, $\lambda \leq \lambda'$ over the directed set Λ. We define an n-cochain group $C^n(\mathcal{G})$ of $\mathcal{G}$ by

$$C^n(\mathcal{G}) = \prod_{\bar{\lambda} \in \Lambda^n} G_{\bar{\lambda}}, n \geq 0,$$

where $G_{\bar{\lambda}} = G_{\lambda_0}$. The coboundary operator $\delta^n : C^{n-1}(\mathcal{G}) \to C^n(\mathcal{G}), n \geq 1$, is defined by

$$(\delta^n y)_{\bar{\lambda}} = (-1)^n [g_{\lambda_0\lambda_1}(y_{\bar{\lambda}_0}) + \sum_{j=1}^{n}(-1)^j y_{\bar{\lambda}_j}],$$

where $y \in C^{n-1}(\mathcal{G})$. For $n = 0$, we put $\delta^0 = 0 : 0 \to C^0(\mathcal{G})$. Then we have a cochain complex

$$(C^*(\mathcal{G}), \delta) : 0 \longrightarrow C^0(\mathcal{G}) \xrightarrow{\delta^1} C^1(\mathcal{G}) \longrightarrow \cdots \longrightarrow C^{n-1}(\mathcal{G}) \xrightarrow{\delta^n} C^n(\mathcal{G}) \longrightarrow \cdots.$$

The n-th *derived group* $\varprojlim_{\lambda}^{(n)}(\mathcal{G})$ of the inverse system $\mathcal{G}$ is defined by the cohomology group of this cochain complex $(C^*(\mathcal{G}), \delta)$.

Consider the following two exact sequence [5];

(3-1)

$$\cdots \to \varprojlim_{\lambda}{}^{(r)}(H_{p+r}(\mathcal{X}; G)) \to \bar{H}_p^{s(r)}(\mathcal{X}; G) \to \bar{H}_p^{s(r-1)}(\mathcal{X}; G) \to \varprojlim_{\lambda}{}^{(r+1)}(H_{p+r}(\mathcal{X}; G)) \to \cdots$$

and

(3-2)

$$0 \longrightarrow \varprojlim_{r}{}^{(1)}(\bar{H}_{p+1}^{s(r)}(\mathcal{X}; G), j) \longrightarrow \bar{H}_p(\mathcal{X}; G) \longrightarrow \varprojlim_{r}(\bar{H}_p^{s(r)}(\mathcal{X}; G), j) \longrightarrow 0.$$

PROPOSITION 3.1. *[6] If the inverse system* $(\bar{H}_*^{s(r)}(\mathcal{X}; G), j)$ *has the Mittag-Leffler property, then*

$$\varprojlim_{\lambda}{}^{(1)}(\bar{H}_*^{s(r)}(\mathcal{X}; G), j) = 0. \quad \square$$

THEOREM 3.2. *Let* $\mathcal{X} = (X_\lambda, f_{\lambda\lambda'}, \Lambda)$ *be an inverse system of CW-spaces with* $\sup\{\dim(X_\lambda) | \lambda \in \Lambda\} = m$*. Then the map*

$$\bar{H}_p(\mathcal{X}; G) \to \bar{H}_p^{s(r)}(\mathcal{X}; G)$$

is an isomorphism.

Proof. Since $X_\lambda, \lambda \in \Lambda$ has a dimension $\leq m$, we have

$$H_{p+r}(X_\lambda; G) = 0.$$

for each $\lambda \in \Lambda$ and $p + r > m$. Thus

$$H_{p+r}(\mathcal{X}; G) = (H_{p+r}(X_\lambda; G), \tilde{f}_{\lambda\lambda'}, \Lambda) = 0$$

and

$$\lim_{\overleftarrow{\lambda}}{}^{(r)}(H_{p+r}(\mathcal{X}; G)) = 0$$

for all $r \geq 0$ and $p + r > m$. By (3.1), we have an isomorphism

$$\bar{H}_p^{s(r)}(\mathcal{X}; G) \to \bar{H}_p^{s(r-1)}(\mathcal{X}; G).$$

Hence the inverse system $(\bar{H}_p^{s(r)}(\mathcal{X}; G), j)$ has the Mittag-Leffler property. By Proposition 3.1 and (3-2), we have that

$$\lim_{\overleftarrow{r}}{}^{(1)}(\bar{H}_p^{s(r)}(\mathcal{X}; G), j) = 0$$

and

$$\begin{aligned} \bar{H}_p(\mathcal{X}; G) &\cong \lim_{\overleftarrow{r}}(\bar{H}_p^{s(r)}(\mathcal{X}; G), j) \\ &\cong \bar{H}_p^{s(r)}(\mathcal{X}; G). \end{aligned}$$

An inverse system $\mathcal{X} = (X_\lambda, f_{\lambda\lambda'}, \Lambda)$ is said to be *n-connected* if the induced inverse system (or pro-group) $\pi_k(\mathcal{X}, *) = (\pi_k(X_\lambda, *), f_{\lambda\lambda'\sharp}, \Lambda)$, $0 \leq k \leq n$ is trivial.

THEOREM 3.3. *If $\pi_{n+1}(\mathcal{X}, *)$ has the Mittag-Leffler property and $(\mathcal{X}, *)$ is $(n-1)$-connected, then*

$$\pi_n(X, *) \cong \check{H}_n(\mathcal{X}; \mathbb{Z}).$$

Proof. Since the sequence

$$0 \to \varprojlim_{\lambda}{}^{(1)}(\pi_{n+1}(\mathcal{X},*)) \to \pi_n(X,*) \to \varprojlim_{\lambda}(\pi_n(\mathcal{X},*)) \to 0$$

is exact [6] and $\pi_{n+1}(\mathcal{X},*)$ has the Mittag-Leffler property, we have

$$\varprojlim_{\lambda}{}^{(1)}(\pi_{n+1}(\mathcal{X},*)) = 0.$$

Thus

$$\begin{aligned}\pi_n(X,*) &\cong \varprojlim_{\lambda}(\pi_n(\mathcal{X},*)) \\ &\cong \varprojlim_{\lambda}(H_n(\mathcal{X};\mathbb{Z})) \text{ (Hurewicz isomorphism theorem)} \\ &\cong \check{H}_n(\mathcal{X};\mathbb{Z}) \text{ (definition of Čech homology group).}\end{aligned}$$

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