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OBTAINING BOUNDARY TANGENTIAL COMPONENTS OF POTENTIAL MAGNETIC FIELDS BY A VARIATIONAL METHOD

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ABSTRACT

An attempt is made to find the boundary tangential components of potential magnetic fields without constructing solutions in the entire domain. In our procedure, the magnetic energy is expressed as a functional of tangential and normal magnetic fields at the boundary and is minimized by the variational principle. This paper reports a preliminary study on two dimensional potential fields above a plane.

Key Words : magnetic fields — methods: analytical — Sun: magnetic fields

I. INTRODUCTION

The magnetic field plays a central role in the dynamics of the solar atmosphere. The solar magnetic field is generated below the solar surface and is governed by plasma motions in and below the photosphere whereas it dominates the plasma dynamics in the solar corona because the plasma β, the ratio of plasma pressure to magnetic pressure, is far smaller than unity there. Thus the plasma pressure is usually ignored in modeling the coronal dynamics. Since the wave transit time across a coronal magnetic structure is far shorter than the time scale over which the photospheric field profile manifests a remarkable change, the coronal magnetic field is also considered as quasi-static except when an abrupt change of state sets in. Most studies on solar atmospheric phenomena have thus been based on force-free models of coronal magnetic fields.

However, constructing a force-free magnetic field based on photospheric boundary conditions is a formidable task and requires much numerical work (e.g., Mikić *et al.* 1989). A linear force-free field, in which the ratio of current density to magnetic field is constant everywhere, can be handled quasi-analytically, but the fact that the magnetic energy of a linear force-free field diverges in an infinite domain sets a limitation on its application to real solar magnetic fields (Low 1994). The simplest case of force-free fields is a potential field and the solution can be easily obtained by a numerical computation or quasi-analytically by a base function expansion.

Boundary value problems seeking force-free solutions can be formulated in two different ways. First, the tangential components of the magnetic field can be given as well as the normal component at the boundary. In this type of problem, usually called BVP1 (boundary value problem 1), the field line connectivity can be known only after the solution is obtained. In another type of problem, usually called BVP2, the positions of field line footpoints are given in addition to the normal magnetic field at the boundary. The tangential components come out as a part of the solution. From the view point of observers, BVP1s are more relevant than BVP2 because all three components of the magnetic field can be obtained from vector magnetograms whereas to theorists, BVP2s are more interesting in that we can follow the evolution of an active region while footpoints are moving. In a potential field problem, only the normal component of the magnetic field is specified and the tangential components are part of the solution.

The magnetic energy of a system in force-free equilibrium can be obtained when all three components of the magnetic field is known at the boundary of the system (e.g., Aly 1984). The tangential components of the magnetic field also contain information about current sheets in the global field configuration (Low and Wolfson 1988). However, theoretical analysis based only on boundary information has been hindered in BVP2s or potential field problems because it requires no less work than finding a global solution.

A magnetohydrostatic (MHS) equilibrium is the state of energy extremum with a boundary condition on the normal magnetic field and some other constraints on the field line connectivity. Thus the equations describing MHS equilibria can be derived from the variational principle (e.g., Courant and Hilbert 1953 for potential field problems, Low 1978 for force-free fields, Chodura and Schlüter 1981 for general MHS equilibria). Some methods of finding MHS equilibria employ variational procedures (e.g., Chodura and Schlüter 1981, Choe and Lee 1996), but they still seek global solutions. If our interest lies in the boundary tangential field components only, it is not necessary to extremize the energy varying the field in the whole domain. Since the energy of an equilibrium can be expressed as a functional of normal and tangential field components at the boundary, we may be able to find the tangential field components extremizing the system energy. It is rather surprising that this kind of attempt has not been reported so far.

Transforming the boundary condition at infinity into a constraint in the variational procedure may have been an obstacle in this approach as will be seen in the following sections.

In this paper, we try to find the boundary tangential magnetic field which minimizes the magnetic energy of the system. In Section 2, generals on the magnetic energy of force-free fields are presented. A specific investigation of 2D potential fields is performed in Section 3. Section 4 gives a summary and discussion.

II. ENERGY OF FORCE-FREE MAGNETIC FIELDS

In this section, we present a mathematical formulation of the force-free field and its energy, mainly following Aly (1984). A force-free magnetic field is described by

$$\mathbf{J} \times \mathbf{B} = 0 , \tag{1}$$

and

$$\nabla \cdot \mathbf{B} = 0 , \tag{2}$$

in which $\mathbf{B}$ is the magnetic field and $\mathbf{J} = \nabla \times \mathbf{B}$ the current density in rationalized electromagnetic units. Equation (1) means that $\mathbf{J}$ is parallel to $\mathbf{B}$, i.e.,

$$\mathbf{J} = \alpha \mathbf{B} . \tag{3}$$

Equation (1) can also be written as

$$\nabla \cdot \mathsf{T} = 0 , \tag{4}$$

in which T is the Maxwell stress tensor given by

$$\mathsf{T} = -\frac{B^2}{2}\mathsf{I} + \mathbf{BB} . \tag{5}$$

The magnetic energy $W(V)$ of a force-free field contained in volume V can be expressed as

$$2W(V) = \int_V \mathbf{B}\cdot\mathbf{B}\, dV = \int_V \alpha \mathbf{A}\cdot\mathbf{B}\, dV + \int_{\partial V} (\mathbf{B}\times\mathbf{A})\cdot\hat{\mathbf{n}}\, d\sigma , \tag{6}$$

in which $\mathbf{A}$ is the magnetic vector potential, ∂V is the boundary of volume V, and $\hat{\mathbf{n}}$ is the unit vector interior normal to ∂V. In deriving equation (6), the vector identity

$$\mathbf{B} \cdot \nabla \times \mathbf{A} = \mathbf{A} \cdot \nabla \times \mathbf{B} - \nabla \cdot (\mathbf{B} \times \mathbf{A}) \tag{7}$$

is used along with equation (3). Another expression of the magnetic energy of a force-free field is obtained from tensor virial equations (Aly 1984);

$$W(V) = \int_V \frac{B^2}{2} dV = \int_{\partial V} \left[B_n (\mathbf{B} \cdot \mathbf{r}) - \frac{B^2}{2} (\mathbf{r} \cdot \mathbf{n}) \right] d\sigma . \tag{8}$$

This equation allows us to compute the magnetic energy from the boundary value of the magnetic field and it finds the most practical use when a domain outside a sphere is concerned. In this paper, our investigation is limited to an unbounded domain above a plane due to the tractability of the problem.

Now we further confine our interest to 2.5D force-free fields in a half-space above a plane. An arcade-like field configuration along a polarity inversion line may be an object of our 2.5D models. To handle this field geometry, a Cartesian coordinate system is employed, in which the x-z plane is taken to correspond to the solar surface, the y-axis is the vertical axis and the z-axis lies on a model polarity inversion line along which all physical quantities are assumed to be invariant. The magnetic field can then be described by

$$\mathbf{B} = \nabla A \times \hat{\mathbf{z}} + B_z \hat{\mathbf{z}} , \tag{9}$$

where A is the z-component of the magnetic vector potential which is constant along individual field lines. Setting $A(x \to \pm\infty) = 0$, we see another meaning of $A(x, y = 0)$, the boundary normal flux, i.e.,

$$A(x) = -\int_{-\infty}^{x} B_y dx = \int_x^{\infty} B_y dx . \tag{10}$$

From equations (1) and (2), it can be shown that

$$\mathbf{B} \cdot \nabla B_z = 0 , \tag{11}$$

$$\mathbf{B} \cdot \nabla J_z = 0 , \tag{12}$$

which implies that B_z and J_z are respectively a function of A only. The toroidal current density can be written as

$$J_z(A) = -\nabla^2 A = \frac{d}{dA}\left(\frac{B_z^2(A)}{2}\right) . \tag{13}$$

This so-called Grad-Shafranov equation is an alternative expression of the equilibrium condition represented by equation (1). The distance in the z-direction between two footpoints of a field line $\zeta(A)$ is given by

$$\zeta(A) = \int_A \frac{B_z}{|\nabla A|} ds_p , \tag{14}$$

where the line integral is performed along the field line represented by A. In a 2.5D Cartesian geometry, the meaningful magnetic energy of the system is the one contained in a semi-infinite slab with unit depth in the z-direction, i.e.,

$$W = \int_{y=0}^{\infty} \int_{x=-\infty}^{\infty} \frac{B^2}{2} dx\, dy . \tag{15}$$

In this study, we only deal with magnetic fields whose energy as well as flux is finite. For this condition to hold, the asymptotic behavior of the field at infinity should be such that

$$\lim_{r\to\infty} B^2 r^2 = 0 . \tag{16}$$

Under this assumption, equation (6) yields the magnetic energy of a 2.5D force-free field

$$2W = -\int_{-\infty}^{\infty} AB_x\,dx - \frac{1}{2}\int_{-\infty}^{\infty} \mathrm{sign}(B_y)\,A\,\zeta\,\frac{\partial B_z}{\partial x}\,dx + \frac{1}{2}\int_{-\infty}^{\infty} |B_y|\,B_z\,\zeta\,dx\ . \tag{17}$$

Here the first and second term represent the poloidal energy and the third term is the toroidal energy. Since the force-free field without shear ζ is nothing but a potential field, the first term is the energy of the potential field with a given boundary normal field distribution. It should be noted that all the integrals in equation (17) are to be performed along the x-axis and all the variables in the integrands are functions of x at $y = 0$. The reason why we take x as the independent variable rather than A is that one value of A corresponds to two (in a single flux system) or more (in a multiple flux system) values of x on the x-axis. In a BVP2 type problem, in which B_y and ζ are given, the tangential field components B_z and B_x may be determined by extremizing the magnetic energy. However, we have not completed a full investigation of this variational problem for general 2.5D force-free fields. In this paper, our report is limited to the simplest case, i.e., the 2D potential field problem.

III. A VARIATIONAL APPROACH TO 2D POTENTIAL MAGNETICFIELDS

A potential field is the minimum energy state of all field configurations having the same boundary normal field distribution. The energy of a 2D potential field in a half-space above a plane is given by (eq. [17])

$$W = -\frac{1}{2}\int_{-\infty}^{\infty} AB_x\,dx\ . \tag{18}$$

Now our task is to find $B_x(x)$ minimizing W. To do this, we must obtain constraints which prescribe the relation between the given normal magnetic field B_y and the tangential magnetic field B_x to be determined. These constraints can be derived from the force-free condition (eq. [4]) and the boundary condition at infinity (eq. [16]). From equation (4), we have

$$\int_V \mathbf{r}^m\,\nabla\cdot\mathbf{T}\,dV = -\int_{\partial V} \hat{\mathbf{n}}\cdot\mathbf{T}\,\mathbf{r}^m\,d\sigma - \int_V \mathbf{T}\cdot\nabla\,\mathbf{r}^m\,dV = 0\ , \tag{19}$$

where $m(\geq 0)$ is an integer and $\hat{\mathbf{n}}$ is the interior normal unit vector. With the boundary condition at infinity given by equation (16), the surface term at infinity in equation (19) vanishes only for $m = 0$ and $m = 1$ in a 2D Cartesian geometry. For $m = 0$, we have two meaningful integral relations between boundary quantities:

$$\int_{-\infty}^{\infty} B_xB_y\,dx = 0\ , \tag{20}$$

$$\int_{-\infty}^{\infty}\left(B_y^2 - \frac{B^2}{2}\right)dx = \frac{1}{2}\int_{-\infty}^{\infty}(B_y^2 - B_x^2)\,dx = 0\ . \tag{21}$$

For $m = 1$, equation (19) comprises four scalar equations, but only two of them are independent of equations (20) and (21):

$$\int_{-\infty}^{\infty} x\,B_xB_y\,dx = 0\ , \tag{22}$$

$$\int_{-\infty}^{\infty} x\left(B_y^2 - \frac{B^2}{2}\right)dx = \frac{1}{2}\int_{-\infty}^{\infty} x\,(B_y^2 - B_x^2)\,dx = 0\ . \tag{23}$$

Equations (20)–(23) constitute the integral constraints on our variational problem and will hereafter be called constraint I, II, III and IV respectively. It is to be noted that equations (20) and (23) are trivial if a symmetry exists across the y-axis. However, no symmetry is assumed in this study.

Minimizing functional W under constraints I–IV is identical to extremizing another functional

$$W^* = \int_{-\infty}^{\infty} K(B_x, x)\,dx\ , \tag{24}$$

where

$$K(B_x, x) = -AB_x + \lambda_1\,B_xB_y + \lambda_2\,(B_y^2 - B_x^2) + \lambda_3\,x\,B_xB_y + \lambda_4\,x\,(B_y^2 - B_x^2)\ , \tag{25}$$

in which λ's are Lagrange multipliers. Since K does not explicitly contain a dB_x/dx term, the Euler equation is simply

$$\frac{\partial K}{\partial B_x} = 0\ . \tag{26}$$

From this we obtain

$$B_x(x) = \frac{-A + \lambda_1\,B_y + \lambda_3\,x\,B_y}{2(\lambda_2 + \lambda_4\,x)}\ . \tag{27}$$

For a potential field with finite energy and flux, field lines tend to lay themselves down parallel to the x-axis as $x \to \pm\infty$. In other words,

$$\lim_{x\to\pm\infty}\frac{B_y}{B_x} = 0\ . \tag{28}$$

From this asymptotic condition, we find that

$$\lambda_4 = 0 \tag{29}$$

and

$$B_x(x) = \frac{-A + \lambda_1\,B_y + \lambda_3\,x\,B_y}{2\,\lambda_2}\ . \tag{30}$$

Substituting this back into constraints I, II and III, we can calculate Lagrange multipliers λ_1, λ_2 and λ_3.

Defining the integrals

$$I_A = \int_{-\infty}^{\infty} B_y^2\, dx \ , \qquad (31)$$

$$I_B = \int_{-\infty}^{\infty} x\, B_y^2\, dx \ , \qquad (32)$$

$$I_C = \int_{-\infty}^{\infty} x^2 B_y^2\, dx \ , \qquad (33)$$

$$I_D = \int_{-\infty}^{\infty} \frac{A^2}{2}\, dx = \int_{-\infty}^{\infty} x\, A\, B_y\, dx \ , \qquad (34)$$

and evaluating the integral

$$\int_{-\infty}^{\infty} A B_y\, dx = 0 \ , \qquad (35)$$

we obtain

$$\lambda_1 = -\frac{I_B\, I_D}{I_A\, I_C - I_B^2} \ , \qquad (36)$$

$$\lambda_2 = \frac{D}{2\,(I_A\, I_C - I_B^2)^{\frac{1}{2}}} \ , \qquad (37)$$

$$\lambda_3 = \frac{I_A\, I_D}{I_A\, I_C - I_B^2} \ . \qquad (38)$$

Our variational problem is thus solved. For a case of symmetry across the y-axis, $I_B = 0$ and λ_1=0. As can be seen in equation (30), λ_2 determines the magnetic field at the origin which is located on a polarity inversion line, and

$$B_x(0,0) = -\frac{A(0,0)}{2\,\lambda_2} \ . \qquad (39)$$

Far enough away from the origin on the x-axis, field lines tend to radiate outward. For this tendency, λ_3 is responsible.

To demonstrate the validity of our method, we construct the tangential field B_x for a dipole flux distribution given by

$$A(x,0) = \frac{1}{x^2+1} \ . \qquad (40)$$

The normal magnetic field for this flux profile is

$$B_y(x,0) = \frac{2x}{(x^2+1)^2} \ . \qquad (41)$$

Evaluation of the integrals defined in equations (31), (32) and (34) yields

$$I_A = I_C = I_D = \frac{\pi}{2} \ ,$$

from which we obtain

$$\lambda_2 = \frac{1}{2} \quad \text{and} \quad \lambda_3 = 1 \ .$$

Substituting this into equation (30) gives

$$B_x(x,0) = \frac{x^2-1}{(x^2+1)^2} \ . \qquad (42)$$

The potential field solution for the boundary flux distribution given by equation (40) is known to be

$$A(x,y) = \frac{y+1}{x^2+(y+1)^2} \ . \qquad (43)$$

This leads to the same $B_x(x,0)$ profile as is obtained by us in equation (42).

IV. SUMMARY AND DISCUSSION

In this paper, we have presented a new variational method of finding the boundary tangential components of 2D potential magnetic fields. The magnetic energy is expressed as a functional of boundary quantities and is minimized under the constraints derived from the tensor virial equations and the boundary condition at infinity. Applied to a dipole field, the validity of our method is confirmed. However, our method can be used for arbitrary multipole field distributions with more than one polarity inversion line. We only need to locate the origin on one of the inversion lines to apply our procedure.

Since no consideration is given to the field line connectivity in our formulation, our solution is a smooth function of space as long as the boundary normal field profile is continuous. In solar plasmas, the field line connectivity is conserved and even a potential field may have current sheets if no smooth solution exists in a certain topology. In such a case, the boundary tangential magnetic field has discontinuities (Low and Wolfson 1988). For very mildly sheared solar magnetic fields, comparing the vector magnetogram and the computational result will tell us whether a singular current surface is involved or not.

In the future, our investigation will proceed to 2.5D force-free fields in BVP2 type problems. Finding a complete set of necessary constraints is the focal problem in this procedure. Furthermore, an inequality constraint comes up in a sheared field problem because the sign of ζ and B_y determines the sign of B_z. With all this difficulty, this approach is more worth pursuing than global numerical calculations because we can obtain the functional form of the differential flux volume from the B_z profile, which provides us with a definite general information about possibility of the field topology change by magnetic reconnection.

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