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On the Crawford Number and Perturbation bounds of Eigenvalue Problems

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Crawford 수와 고유치문제의 섭동유계에 관하여

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We introduce the Crawford number and codal metric to analyze perturbation bounds of the generalized eigenvalue problem.

일반화 된 고유치문제의 섭동 유계에 관하여 Crawford 수와 codal 측정을 이용하여 분석한다.

Key words : Crawford number, Codal metric, Perturbation, Eigenvalue

I. Introduction

Let A and B be matrices of order n with complex elements. By the eigenvalue problem of A with respect to B, we mean the problem of determining the set of all λ for which the equation $Ax=\lambda Bx$ has a nontrivial solution. In this case λ is called an eigenvalue of A with respect to B and x an eigenvector corresponding to the eigenvalue λ. The generalized eigenvalue problem $Ax=\lambda Bx$ arises in the theory of systems of ordinary differential equations with constant coefficients and has important physical applications. We shall be concerned with the anlalytic problem of determining perturbation bounds for the eigenvalues and eigenvectors of $Ax=\lambda Bx$ where A and B are Hermitian matrices of order n.

Whenever A and B have a common null space except zero, the characteristic polynomial $\det(A-\lambda B)$ is identically zero. That is, every number is an eigenvalue of $Ax=\lambda Bx$. In the case that B is singular, the characteristic polynomial is of degree less than n. That is, $Ax=\lambda Bx$ has an infinite eigenvalue and the characteristic polynomial of the reciprocal problem $Bx=\mu Ax$ has a zero at the origin equal in multiplicity to the defct k in the charateristic polynomial $\det(A-\lambda B)$. A corresponding eigenvector for both problems is natuarally a null vector of B. If B is perturbed slightly, the eigenvalues of $Ax=\lambda Bx$ will generally contatin k large eigenvalues that become infinite as the perturbation is reduced to zero. If B is nearly singular, the nearly infinite eigenvalues of $Ax=\lambda Bx$ are extremely sensitive to perturbations in B and can not be calculated accurately without high precision arithmetic.

It has been known that even if the large eigenvalues of the problem usually undergo large perturbations, their reciprocals will undergo only small perturbations. This suggests that the usual Euclidean metric on the line is not appropriate for reporting the sizes of the perturbations in the eigenvalues. So we introduce the concepts of Crawford number and chodal metric in section 2. In the section 3 we will give some facts for perturbation bounds of eigenvalues. Finally we will conclude with some remarks in section 4.

II. On the Crawford number

The idea of the field of values of a complex n by n matrix C was introduced by Toeplitz in 1918. It is the set of complex numbers defined by

$$W(C)=\{x^*Cx \mid \|x\|=1, x\in \mathbb{C}^n\}.$$

Since it is closed and bounded, it is compact. It is also a connected set. Toeplitz showed in 1918 that W has a convex outer boundary, and Hausdorff proved in 1919 that W itself is convex. Kipperhahn also showed in 1951 that W is described as the convex hull of a certain algebraic curve of degree n obtainable from C.

It was well known that every matrix C in $\mathbb{C}^{n\times n}$ is decomposed uniquely by

$$C=A+iB$$

where A and B in $\mathbb{C}^{n\times n}$ are Hermitian. Hence we assume A and B are Hermitian and consider the set $W(A,B)$ defined by

$$W(A,B)=\{x^*(A+iB)x \mid \|x\|=1, x\in \mathbb{C}^n\}$$

For real symmetric matrices A and B, we define

$$R(A,B)=\{x^T(A+iB)x \mid \|x\|=1, x\in \mathbb{R}^n\}$$

Then we have that

$$R(A,B)\subset W(A,B).$$

Brickman[1] showed that for any $n\geq 3$,

$$R(A,B) = W(A,B).$$

Thus $R(A,B)$ is convex for any $n\geq 3$, but it is not true for $n=2$.

Althought we can have $R\neq W$ for $n=2$, it is now clear that W is the convex hull of R in any case[1]. We consider the generalized eigenvalue problem $Ax=\lambda Bx$,

where A and B are Hermitian. We give a definition of the Crawford number for A and B.

Definition. The eigenvalue problem $Ax=\lambda Bx$ is said to be definite if

$$c(A,B)\equiv\inf\{\,|w| : w\in W(A,B)\}>0,$$

which is called the Crawford number for A and B.

We consider the transformation (A_φ, B_φ) of (A, B),

$$\begin{pmatrix} A_\varphi \\ B_\varphi \end{pmatrix} = \begin{pmatrix} \cos(\varphi) & -\sin(\varphi) \\ \sin(\varphi) & \cos(\varphi) \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}.$$

Let X be a matrix for which X^*AX and X^*BX are diagonal.

Since

$$X^*A_\varphi X = \cos(\varphi)X^*AX - \sin(\varphi)X^*BX$$

and

$$X^*B_\varphi X = \sin(\varphi)X^*AX + \cos(\varphi)X^*BX$$

the matrix also diagonalizes A_φ and B_φ. It follows that the problem

$$Ax = \lambda Bx$$

is equivalent to the problem

$$A_\varphi x = \lambda B_\varphi x.$$

So we want to take φ so that we have an equivalent promble to $Ax = \lambda Bx$, in which B_φ is positive definite.

Theorem 1. If the problem $Ax = \lambda Bx$ is definite, there is a real constant φ such that B_φ is positive definite and

$$c(A, B) = \lambda_{\min}(B_\varphi),$$

where $\lambda_{\min}(B_\varphi)$ denotes the smallest eigenvalue of B_φ[4].

This implies that the Hermitian matrix $B_\varphi^{-\frac{1}{2}} A_\varphi B_\varphi^{-\frac{1}{2}}$ exists and can be diagonalized by a unitary matrix Y. It follows that $B_\varphi^{-\frac{1}{2}} Y$ diagonalizes both A_φ and B_φ, and thus A and B. In other words, if the problem $Ax = \lambda Bx$ is definite, then there is a nonsingular matrix X such that X^*AX and X^*BX are both diagonal. If we set

$$X^*AX = diag(\mu_1, \mu_2, \cdots, \mu_n)$$

and

$$X^*BX = diag(v_1, v_2, \cdots, v_n),$$

then the eigenvalues of $Ax = \lambda Bx$ are given by

$$\lambda_i = \frac{\mu_i}{v_i}.$$

This admits the possibility of an infinite eigenvalue when $v_i = 0$. However the indeterminant case $\mu_i = v_i = 0$ can not occur in a definite problem because if they can, there is a corresponding eigenvector x_i which is the i-th column of X such that

$$x_i^*(A + iB)x_i = 0.$$

This is a contradiction to $c(A, B) > 0$.

For convenience we assume the perturbations are Hermitian. Then we have a following theorem[4].

Theorem 2.

$$c(A+E, B+F) \\ \geq c(A, B) - (\| E \|^2 + \| F \|^2)^{\frac{1}{2}}$$

This implies that if the perturbations of A and B are small enough so that

$$c(A, B) - (\| E \|^2 - \| F \|^2)^{\frac{1}{2}}$$

is positive, then the problem

$$(A+E)x = \lambda(B+F)x$$

is also definite.

III. On perturbation bounds

In this section we will summarize some results for the definite generalized eienvalue problem. For the Hermitian eigenvalue problem $Ax=\lambda x$, it is well known that if the eigenvalues are ordered so that

$$\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n,$$

and those of the perturbed problem $\tilde{A}\tilde{x}=\tilde{\lambda}\tilde{x}$ are ordered so that

$$\tilde{\lambda}_1 \le \tilde{\lambda}_2 \le \cdots \le \tilde{\lambda}_n$$

then

$$|\lambda_i - \tilde{\lambda}_i| \le \| A-\tilde{A} \|$$

for $i=1,2,\cdots,n$. Let the problem $Ax=\lambda Bx$ be definite, and let

$$\| E \|^2 + \| F \|^2 < c(A, B).$$

Then theorem 2 says the perturbed problem $\tilde{A}\tilde{x}=\tilde{\lambda}\tilde{B}\tilde{x}$ is also definite.

Let $Ax_i=\lambda_i Bx_i$, where $x_i \ne 0$. We define the angle associated with λ_i to be

$$\theta_i = \theta(x_i^* A x_i, x_i^* B x_i)$$

and assume that

$$\theta \le \theta_2 \le \cdots \le \theta_n.$$

If B is positive definite, this corresponds to the natural ordering

$$\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$$

of the eigenvalues.

Theorem 3. Let the problem $Ax = \lambda Bx$ be definite. Then

$$\begin{aligned}\theta_i &= \min_{\dim X = i}\ \max_{\substack{x\in X\\ x\ne 0}} \theta(x^T Ax, x^T Bx)\\ &= \max_{\dim X = n-i+1}\ \min_{\substack{x\in X\\ x\ne 0}} \theta(x^T Ax, x^T Bx)\end{aligned}$$

As a consequence of theorem 3, a number of separation theorems for the eigenvalues of symmetric matrices generalize to the definite problem. For convenience, if $\hat{A}$ and $\hat{B}$ are principal submatrices of A and B of order $n-1$, then the eigenvalues

$$\hat{\lambda}_1 \le \hat{\lambda}_2 \le \cdots \le \hat{\lambda}_{n-1}$$

of the promblem $\hat{A}x=\hat{\lambda}\hat{B}x$ satisfy

$$\theta_1 \le \hat{\theta}_1 \le \theta_2 \le \cdots \le \hat{\theta}_{n-1} \le \theta_n.$$

Theorem 4. Let $Ax=\lambda Bx$ be definite and let the eigenvalues of $Ax=\lambda Bx$ be ordered so that $\theta_1 \le \theta_2 \le \cdots \le \theta_n$.

Let $e = \{ \| E \|^2 + \| F \|^2 \}^{\frac{1}{2}}$

and assume that $\varepsilon < c(A, B)$ so that $\tilde{A}\tilde{x}=\tilde{\lambda}\tilde{B}\tilde{x}$ is definite, where $\tilde{A}=A+E$ and $\tilde{B}=B+F$. Let the eigenvalues of $\tilde{A}\tilde{x}=\tilde{\lambda}\tilde{B}\tilde{x}$ be ordered so that $\tilde{\theta}_1 \le \tilde{\theta}_2 \le \cdots \le \tilde{\theta}_n$

Then $|\theta_i - \tilde{\theta}_i| \le \sin^{-1}\dfrac{\varepsilon}{c(A, B)}$.

The bound in the theorem 4 implies a bound in the chodal metric.

Definition. Let $\lambda = \frac{\mu}{v}$ and $\tilde{\lambda} = \frac{\tilde{\mu}}{\tilde{v}}$. Then the chordal distance between λ and $\tilde{\lambda}$ is defined by

$$\chi(\lambda, \tilde{\lambda}) = \frac{|\mu\tilde{v} - \tilde{\mu}v|}{\sqrt{\mu^2+v^2}\sqrt{\tilde{\mu}^2+\tilde{v}^2}}.$$

It follows from

$$\sin(\alpha-\beta) = \sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta)$$

that

$$\chi(\lambda, \tilde{\lambda}) = \sin|\theta(\mu, v) - \theta(\tilde{\mu}, \tilde{v})|.$$

Hence

$$\chi(\lambda_i, \tilde{\lambda}_i) \le \frac{e}{c(A, B)}.$$

This theorem also implies the classical bounds for the Hermitian eigenvalue problem. Let $B = I$, and let $\lambda_i(\tau) = \frac{\lambda_i}{\tau}$ be the eigenvalue of the problem

$$Ax = \lambda_i(\tau)(\tau I)x,$$

where λ_i is an eigenvalue of $Ax = \lambda x$, which is not depend on τ. Then

$$c(A, \tau I) = \inf_{\|x\|=1}\sqrt{[(x^*Ax)^2+\tau^2]} \approx \tau + O(1)$$

if τ is large. Hence

$$|\theta_i(\tau) - \tilde{\theta}_i(\tau)| = |\lambda_i(\tau) - \tilde{\lambda}_i(\tau)| + O(\frac{1}{\tau^2})$$

Consequently,

$$|\lambda_i - \tilde{\lambda}_i| = |\lambda_i(\tau) - \tilde{\lambda}_i(\tau)|\,\tau$$
$$\le |\theta_i(\tau) - \tilde{\theta}_i(\tau)|\,\tau \le \varepsilon + O(\frac{1}{\tau})$$

which gives the classical result, as $\tau \to \infty$,

$$|\lambda_i - \tilde{\lambda}_i| \le \|A - \tilde{A}\|.$$

Crawford[3] showed the following bounds.

Theorem 5. *Let $\tilde{A} = A + E$ and $\tilde{B} = B + F$, where A, B, E and $F \in \mathbb{R}^{n\times n}$ are symmetric with B and $B+F$ positive definite. Then*

$$|\lambda_n - \tilde{\lambda}_n|$$
$$\le \|(B+F)^{-1}\|(\|E\|^2 + \|F\|^2)^{\frac{1}{2}}.$$

Notices that this is expressed in terms of chordal metric as

$$\chi[(\lambda_n, 1), (\tilde{\lambda}_n, 1)]$$
$$\le \|(B+F)^{-1}\|(\|E\|^2 + \|F\|^2)^{\frac{1}{2}}.$$

IV. Concluding Remarks

Let $Ax = \lambda Bx$ and $B > 0$. Then it is equivalent to the problem

$$B^{-\frac{1}{2}}AB^{-\frac{1}{2}}y = \lambda y,$$

which has the real eigenvalues. It is clear that the positiveness of B implies the problem $Ax = \lambda Bx$ is definite. But the converse is not true. Since the set $\{x : \|x\| = 1\}$ is closed and bounded, it follows that $c(A, B) > 0$ if and only if $x^*(A+iB)x > 0$, for any x with $\|x\| = 1$. Hence we conclude that $A \ge 0 (or\ B \ge 0)$ and $\ker B \cap \ker A = \{0\}$ if and only if the problem $Ax = \lambda Bx$ is definite. So the paper[4] extends the perturbation bounds to the matrix B which is singular.

Consider the transformation

$$\begin{pmatrix} A_\varphi \\ B_\varphi \end{pmatrix} = \begin{pmatrix} \cos(\varphi) & -\sin(\varphi) \\ \sin(\varphi) & \cos(\varphi) \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}.$$

If (λ, u) is an eigenpair of $Ax = \lambda Bx$, then (μ, u) is an eigenpair of $A_\varphi x = \lambda B_\varphi x$, where

$$\mu = \frac{\lambda\cos(\varphi) - \sin(\varphi)}{\lambda\sin(\varphi) + \cos(\varphi)}$$

Conversely if μ is an eigenvalue of $A_\varphi x = \mu B_\varphi x$, then λ is an eigenpair of $Ax = \lambda Bx$, where λ is

$$\lambda = \frac{\mu\cos(\varphi) + \sin(\varphi)}{\cos(\varphi) - \mu\sin(\varphi)}$$

Consequently the problem $Ax = \lambda Bx$ with $B \geq 0$ is changed to the equivalent problem $A'x = \lambda B'x$ with $B' > 0$ so that we can find the perturbation bounds from the generalized eigenvalue problem with positive definite Gram matrix.

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