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Complete Convergence in a Banach Space

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바나하 공간에서의 완전 수렴성

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Let $\{X_{ni}, 1\le i\le n, n\ge 1\}$ be an array of rowwise independent B-valued random variables which is uniformly bounded by a random variable X satisfying $E|X|^{2p} < \infty$ for some $p\ge 1$. Let $\{a_{ni}, 1\le i\le n, n\ge 1\}$ be an array of constants. Under some auxiliary conditions on $\{a_{ni}\}$, it is shown that $\sum_{i=1}^{n} a_{ni}X_{ni} \to 0$ in probability if and only if $\sum_{i=1}^{n} a_{ni}X_{ni}$ converges completely to 0.

$\{X_{ni}, 1\le i\le n, n\ge 1\}$은 2p차 적률을 갖는 적당한 확률변수 X에 의해서 유계된 바나하 공간상의 값을 갖는 확률변수 열이다. 상수 열 $\{a_{ni}, 1\le i\le n, n\ge 1\}$에 적당한 조건을 부여할 때 $\sum_{i=1}^{n} a_{ni}X_{ni}$가 0에 확률적으로 수렴할 조건과 완전수렴할 조건은 서로 동치이다.

Key words : Banach space, B-valued random variables, complete convergence

I . Introduction

Let (B, || ||) be a real separable Banach space. Let $\{X_{ni}, 1\le i\le n, n\ge 1\}$ be an array of rowwise independent B-valued random variables and let $\{a_{ni}, 1\le i\le n, n\ge 1\}$ be an array of constants. An array $\{X_{ni}\}$ is said to be uniformly bounded by a random variable X if for all n and i and each $t>0$,

$$P(\|X_{ni}\| > t)\le P(|X| > t).$$

Recently, Sung[2] obtained some complete convergence theorems for arrays of rowwise independent B-valued random variables. The main theorem of Sung is as follows.

Theorem 1.1. Let $\{X_{ni}, 1\le i\le n, n\ge 1\}$ be an array of rowwise independent B-valued random variables which is uniformly bounded by a random variable X satisfying $E|X|^{2p}<\infty$ for some $p\geq 1$. Let $\{a_{ni}, 1\leq i\leq n, n\geq 1\}$ be an array of real numbers satisfying

$$\max_{1\leq i\leq n}|a_{ni}| = O(1/n^{1/p}) \qquad (1)$$

for some $p\geq 1$, and

$$\sum_{i=1}^{n} a_{ni}^2 = o(1/\log n). \qquad (2)$$

Assume that

$$\sum_{i=1}^{n} a_{ni}X_{ni} \to 0 \text{ in probability.}$$

Then $\sum_{i=1}^{n} a_{ni}X_{ni}$ converges completely to 0.

Note that the above theorem generalized and improved that of Wang et al[3].

In this paper, we prove Theorem 1.1 under some weaker condition than (2).

II. Main Result

To prove our main result, we will need the following lemmas. Lemma 2.1 and Lemma 2.2 come from Sung[2] and de Acosta[1], respectively.

Lemma 2.1. Let $X_1, \cdots, X_n$ be independent B-valued random variables such that $\|X_i\|\leq b_i$ for $1\leq i\leq n$, and let

$S_n = \sum_{i=1}^{n} X_i$. Then for any t>0,

$$E[\exp(t\|S_n\|)] \leq \exp\{tE\|S_n\| + 2t^2\sum_{i=1}^{n} e^{2tb_i}E\|X_i\|^2\}.$$

Lemma 2.2. Let $X_1, \cdots, X_n$ be independent B-valued random variables with $E\|X_i\|^r<\infty$ for $i=1,\cdots,n$ and $1\leq r\leq 2$.

Then $E|\,\|S_n\| - E\|S_n\|\,|^r \leq C_r\sum_{i=1}^{n} E\|X_i\|^r$,

where $S_n = \sum_{i=1}^{n} X_i$, and C_r is a positive constant depending only on r; if r=2 then it is possible to take $C_2 = 4$.

Now we state and prove our main theorem.

Theorem 2.3. Let $\{X_{ni}, 1\leq i\leq n, n\geq 1\}$ be an array of rowwise independent B-valued random variables which is uniformly bounded by a random variable X satisfying $E|X|^{2p}<\infty$ for some $p\geq 1$. Let $\{a_{ni}, 1\leq i\leq n, n\geq 1\}$ be an array of real numbers satisfying (1) and

$$\sum_{n=1}^{\infty}\exp\left(-\frac{t}{A_n}\right)<\infty \qquad (3)$$

for each $t>0$, where $A_n = \sum_{i=1}^{n} a_{ni}^2$.

Assume that

$$\sum_{i=1}^{n} a_{ni}X_{ni} \to 0 \text{ in probability.} \qquad (4)$$

Then $\sum_{i=1}^{n} a_{ni}X_{ni}$ converges completely to 0.

Proof. The proof is based on certain ideas of Sung[2]. Without loss of generality, we may assume that $\max_{1\leq i\leq n}|a_{ni}|\leq 1/n^{1/p}$ and $E|X|^{2p} = 1$. Let $\varepsilon>0$ be given. For $p<q<2p$ and N, define

$$X'_{ni} = X_{ni}I(\|X_{ni}\|\leq n^{1/q}),$$
$$X''_{ni} = X_{ni}I(\|X_{ni}\|>\varepsilon n^{1/p}/N),$$
$$X'''_{ni} = X_{ni}I(n^{1/q}<\|X_{ni}\|\leq \varepsilon n^{1/p}/N),$$

$$T'_n = \sum_{i=1}^{n} a_{ni} X'_{ni},\ T''_n = \sum_{i=1}^{n} a_{ni} X''_{ni},$$
$$T'''_n = \sum_{i=1}^{n} a_{ni} X'''_{ni}.$$

To prove the theorem, it is enough to show that

$$\sum_{n=1}^{\infty} P(\|T'_n\| > \varepsilon) < \infty, \tag{5}$$
$$\sum_{n=1}^{\infty} P(\|T''_n\| > \varepsilon) < \infty, \tag{6}$$

and

$$\sum_{n=1}^{\infty} P(\|T'''_n\| > \varepsilon) < \infty, \tag{7}$$

since

$$\sum_{n=1}^{\infty} P(\|\sum_{i=1}^{n} a_{ni} X_{ni}\| > 3\varepsilon) \le \sum_{n=1}^{\infty} [P(\|T'_n\| > \varepsilon) + P(\|T''_n\| > \varepsilon) + P(\|T'''_n\| > \varepsilon)].$$

We establish (5) only, since the proofs of (6) and (7) are the same as the proof of Sung[2]. From Markov's inequality and Lemma 2.1, we have for t>0

$$\begin{aligned} &P(\|T'_n\| > \varepsilon) \\ &\le e^{-t\varepsilon} E[\exp(t\,\|T'_n\|)] \\ &\le \exp\{-t\varepsilon + tE\|T'_n\| + \\ &2t^2 e^{2t\frac{n^{1/q}}{n^{1/p}}} \sum_{i=1}^{n} E\|a_{ni} X'_{ni}\|^2\}. \end{aligned} \tag{8}$$

Let $t = \min\{\frac{\varepsilon}{4e^2 A_n}, n^{\frac{1}{p}-\frac{1}{q}}\}$.

Now we calculate the power of exp in the last expression of (8).

If $\varepsilon/4e^2A_n \le n^{\frac{1}{p}-\frac{1}{q}}$, *i.e.*, $t = \varepsilon/4e^2A_n$, then the power is dominated by

$$\begin{aligned} &-\frac{\varepsilon^2}{4e^2A_n} + \frac{\varepsilon E\|T'_n\|}{4e^2A_n} + \\ &2\left(\frac{\varepsilon}{4e^2A_n}\right)^2 e^2 \sum_{i=1}^{n} E\|a_{ni}X'_{ni}\| \\ &\le -\frac{\varepsilon^2}{4e^2A_n} + \frac{\varepsilon E\|T'_n\|}{4e^2A_n} \\ &+ 2\left(\frac{\varepsilon}{4e^2A_n}\right)^2 e^2 A_n \\ &= -\frac{\varepsilon^2}{8e^2A_n} + \frac{\varepsilon E\|T'_n\|}{4e^2A_n}, \end{aligned}$$

since $\sum_{i=1}^{n} E\|a_{ni}X'_{ni}\| \le A_n$.

If $\varepsilon/4e^2A_n > n^{\frac{1}{p}-\frac{1}{q}}$, *i.e.*, $t = n^{\frac{1}{p}-\frac{1}{q}}$, then the power is dominated by

$$\begin{aligned} &-\varepsilon n^{\frac{1}{p}-\frac{1}{q}} + n^{\frac{1}{p}-\frac{1}{q}} E\|T'_n\| \\ &+ 2n^{2(\frac{1}{p}-\frac{1}{q})} e^2 A_n \\ &\le -\varepsilon n^{\frac{1}{p}-\frac{1}{q}} + n^{\frac{1}{p}-\frac{1}{q}} E\|T'_n\| \\ &+ 2n^{2(\frac{1}{p}-\frac{1}{q})} \frac{e^2\varepsilon}{4e^2 n^{\frac{1}{p}-\frac{1}{q}}} \\ &= -\frac{\varepsilon n^{\frac{1}{p}-\frac{1}{q}}}{2} + n^{\frac{1}{p}-\frac{1}{q}} E\|T'_n\|, \end{aligned}$$

since $A_n \le \varepsilon/4e^2 n^{\frac{1}{p}-\frac{1}{q}}$.

Hence we have that

$$\begin{aligned} &P(\|T'_n\| > \varepsilon) \le \\ &\exp\left(-\frac{\varepsilon^2}{8e^2A_n} + \frac{\varepsilon E\|T'_n\|}{4e^2A_n}\right) + \\ &\exp\left(-\frac{\varepsilon n^{\frac{1}{p}-\frac{1}{q}}}{2} + n^{\frac{1}{p}-\frac{1}{q}} E\|T'_n\|\right). \end{aligned} \tag{9}$$

On the other hand, we have by Lemma 2.2 and (3) that

$$E\left|\ \|\sum_{i=1}^{n} a_{ni}X_{ni}\| - E\|\sum_{i=1}^{n} a_{ni}X_{ni}\|\ \right|^2$$
$$\le 4\sum_{i=1}^{n} E\|a_{ni}X_{ni}\|^2 \le 4E|X|^2 A_n$$
$$\le 4A_n \to 0.$$

Combining this result and (4) gives

$$E\|\sum_{i=1}^{n} a_{ni}X_{ni}\| \to 0.$$

Thus, it follows that

$$E\|T_n\|$$
$$= E\|\sum_{i=1}^{n} a_{ni}(X_{ni} - X_{ni}I(\|X_{ni}\| > n^{1/q}))\|$$
$$\le E\|\sum_{i=1}^{n} a_{ni}X_{ni}\| +$$
$$\frac{1}{n^{1/p}}\sum_{i=1}^{n} E\|X_{ni}\|I(\|X_{ni}\| > n^{1/q})$$
$$\le E\|\sum_{i=1}^{n} a_{ni}X_{ni}\| + \frac{n}{n^{1/p}} E|X|I(|X| > n^{1/q})$$
$$\le E\|\sum_{i=1}^{n} a_{ni}X_{ni}\| + \frac{n}{n^{1/p+(2p-1)/q}} E|X|^{2p}$$
$$\to 0,$$

since

$$\frac{1}{p} + \frac{2p-1}{q} - 1 = \left(\frac{1}{p} - \frac{1}{q}\right) + \left(\frac{2p}{q} - 1\right) > 0.$$

Hence $E\|T_n\| < \varepsilon/4$ for n sufficiently large. Thus, we have by (9) that

$$\sum_{n=1}^{\infty} P(\|T_n\| > \varepsilon)$$
$$\le \sum_{n=1}^{\infty}\left[\exp\left(-\frac{\varepsilon^2}{8e^2A_n} + \frac{\varepsilon E\|T_n\|}{4e^2A_n}\right) + \right.$$
$$\left.\exp\left(-\frac{\varepsilon n^{\frac{1}{p}-\frac{1}{q}}}{2} + n^{\frac{1}{p}-\frac{1}{q}}E\|T_n\|\right)\right]$$
$$\le C\sum_{n=1}^{\infty}\left[\exp\left(-\frac{\varepsilon^2}{16e^2A_n}\right) + \right.$$
$$\left.\exp\left(-\frac{\varepsilon n^{\frac{1}{p}-\frac{1}{q}}}{4}\right)\right] < \infty$$

for some constant C>0. The last inequality follows from (3) and the fact that $1/p - 1/q > 0$. Thus (5) is proved.

Remark. Since the condition (2) implies the condition (3), Theorem 2.3 is an extension of Theorem 1.1.

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