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삼차원 공간상에서의 질적인 삼각화에 관한 연구

On Quality Triangulation in Three-Dimensional Space*

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Abstract

This paper deals with the problem of generating a uniform tetrahedral mesh which fills a 3-D space with the tetrahedra which are close to the equilateral tetrahedra as possible. This problem is particularly interesting in finite element modeling where a fat triangulation minimizes the error of an analysis. Fat triangulation is defined as a scheme for generating an equilateral triangulation as possible in a given dimension.

In finite element modeling, there are many algorithms for generating a mesh in 2-D and 3-D. One of the difficulties in generating a mesh in 3-D is that a 3-D object can not be filled with uniform equilateral tetrahedra only regardless of the shape of the boundary. Fat triangulation in 3-D has been proved to be the one which fills a 3-D space with the tetrahedra which are close to the equilateral as possible. Topological and geometrical properties of the fat triangulation and its application to meshing algorithm are investigated.

1. Introduction

Finite element modeling can be explained as a modeling scheme where a 2-D or 3-D objectis decomposed into a set of small elements in order to be applied for an engineering analysis. The set of small elements is called a mesh. This mesh generation is

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important because the error of an analysis can be varied depending on the shape of mesh and the mesh generation is time consuming process. In the analysis, it has been known that the triangulation which is close to an equilateral triangulation is the most desirable [Strang 76]. Mesh generation algorithms have been studied actively since the computer graphics was applied in engineering. With the aid of computer graphics, modeling process of an analysis was able to be examined closely. In 3-D, several algorithms for generating a mesh have been developed [Woo and Thomasma 83, Yerry & Shephard 84, Cavendish 85, Frey & Field 85, Chae 88, Dey, Bajaj & Sugihara 91, Field and Smith 91, Michell & Vavasis 92, Ruppert 93].

Among them, only a few considered the effectiveness of the mesh configuration at the initial mesh generation stage. Furthermore, even for these work, the shape a tetrahedral mesh generated is far from the satisfactory. One major difficulty of 3-D mesh generation is that 3-D space can not be filled with the equilateral tetrahedral mesh only [Field & Frey 85]. By a mathematician, a scheme for generating a uniform triangulation in an n-dimension was published satisfying the condition of equilateral triangulation as possible [Todd 85]. In the following sections, the scheme will be reviewed and the topological as well as geometrical properties of the the triangulation in 3-D are investigated. Also, application of these properties to the mesh generation algorithm is discussed.

2. Fat Triangulation

Todd, in 1985, developed an optimal linear transformation of a certain regular triangulation in an n-dimensional space so that the resulting mesh can be uniform while satisfying the condition of equilateral triangulation as possible. The transformation uses K1 triangulation as a regular triangulation which was developed in his earlier work. In K1 triangulation of an n-dimensional space, an initial vertex is chosen as any point in the integer lattice Z_n. Then each subsequent vertex is obtained from its predecessor by taking a unit step along a previously unused coordinate direction. The convex hull of the $n+1$ vertices so formed is a particular simplex of K1, e.g. the convex hull of 0, e_1, $e_1 + e_2$, ... ,$e_1 + ... + e_n$. Todd also introduced a matrix A which transforms a K1 triangulation into a triangulation which is close to an equilateral triangulation as possible and it was proved that the matrix A is optimal.

The transform matrix A is,

$$A = \overline{A} \equiv I-(n+1+\sqrt{n+1})^{-1}ee_T \quad (1)$$

where e denotes the unit vector matrix in a given n-dimension. The triangulation generated after applying A to the K1 triangulation is called a fat triangulation because it is not skewed to any particular directions. In the following section we investigate the properties of the fat triangulation in 2-D and 3-D.

3. Analysis of Fat Triangulation

K1 triangulation and the transform matrix A were defined in n dimensions in order to generate a fat triangulation. K1 triangulation in 3-D can be found if we consider n=3 in the previous section. Figure 1 represents a K1 triangulation in 3-D.

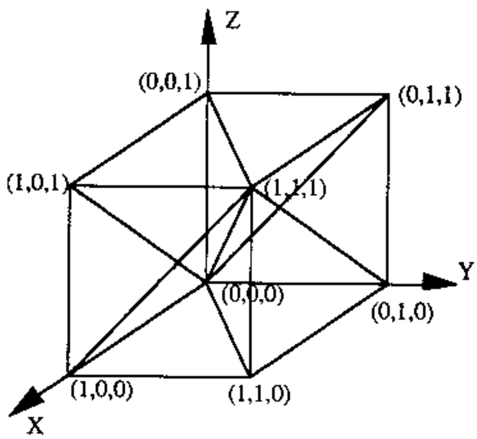

Figure 1. K1 triangulation in 3-D

One interesting property of this triangulation is that the volume of each simplex is identical. For example, the volume of each tetrahedron is 1/6, if the volume of the original cube is 1 in the Figure 1. Figure 2 shows the 6 tetrahedra of the K1 triangulation in 3-D.

In Figure 2, it can be seen that there are two types of tetrahedra and one type is symmetrical to the other. To observe the topological information, we could consider an interior vertex with 8 surrounding tetrahedra.

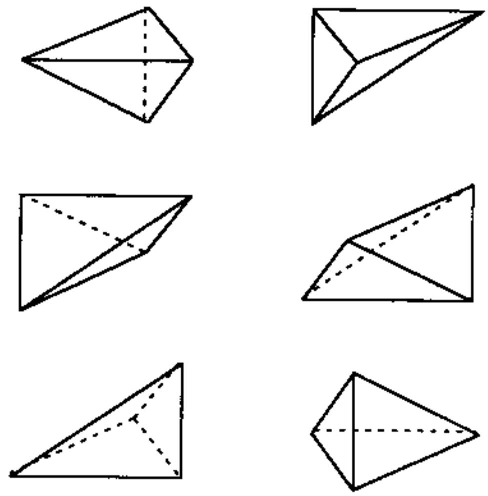

Figure 2. A cube can be decomposed into 6 tetrahedra having same volume

The interior vertex is connected to 14 neighboring vertices and these vertices make 24 tetrahedra. This topological information has an important meaning for generating a tetrahedral mesh in 3-D. This can be explained easily by illustrating the analogy of 2-D. In 2-D object, after generating a mesh, any node is connected to a certain number of edges. If a node is connected to large number of edges, the elements connected to the node will have acute angles which are undesirable in finite element modeling. On the other hand, if a node is connected to a small number of edges, the elements connected to the node will have obtuse angles which are also undesirable. To generate an equilateral triangulation, it can easily be seen that the number of the edges connected to a node should be six since six equilateral triangles can fill up a space around a node in 2-D, when there is no restriction on the boundary of the space, as shown in Figure 3.

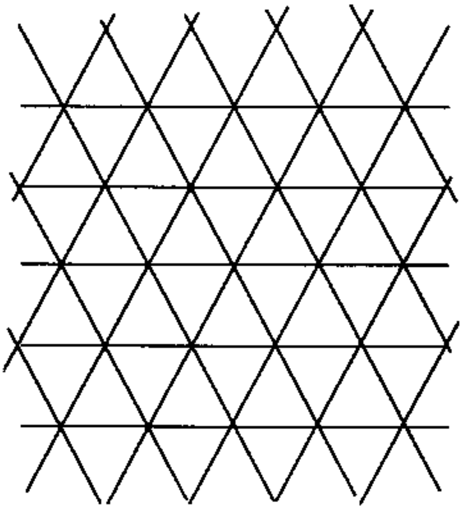

Figure 3. Two dimensional space filled with equilateral triangles

In 3-D, however, any number of equilateral tetrahedra can not fill up a space because an angle defined by adjacent two faces in the equilateral tetrahedron is 72.221 degrees. By examining the fat triangulations, we know the optimal number of edges connected to a vertex to make an equilateral tetrahedral mesh as possible.

In addition to the topological advantage we have discussed thus far, there are also some geometrical advantages for using the fat triangulation in finite element modeling. To see the geometrical advantages, let us see the fat triangulation in 2-D first. We calculate the transform matrix A by substituting n=2 in the equation 1.

$$A = \begin{bmatrix} \frac{2+\sqrt{3}}{3+\sqrt{3}} & -\frac{1}{3+\sqrt{3}} \\ -\frac{1}{3+\sqrt{3}} & \frac{2+\sqrt{3}}{3+\sqrt{3}} \end{bmatrix}$$

This matrix converts a K1 triangulation into a fat triangulation which is perfect equilateral triangulation as in Figure 4.

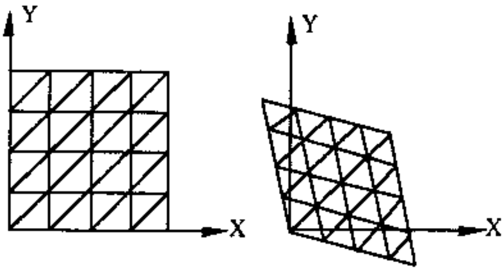

Figure 4. Linear transformation of K1 triangulation in 2-D

By using the similar method, the transform matrix A in 3-D becomes,

$$A = \begin{bmatrix} \frac{5}{6} & \frac{1}{6} & \frac{1}{6} \\ -\frac{1}{6} & \frac{5}{6} & \frac{1}{6} \\ -\frac{1}{6} & \frac{1}{6} & \frac{5}{6} \end{bmatrix}$$

To find the fat triangulation in 3-D, we multiply the vertices of a unit cube which is decomposed by K1 triangulation into this matrix. For example, one tetrahedron in the K1 triangulation is transformed into a fat triangulation in the following way.

$$\begin{bmatrix} \frac{5}{6} & -\frac{1}{6} & -\frac{1}{6} \\ -\frac{1}{6} & \frac{5}{6} & -\frac{1}{6} \\ -\frac{1}{6} & -\frac{1}{6} & \frac{5}{6} \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & \frac{5}{6} & \frac{2}{3} & \frac{1}{2} \\ 0 & -\frac{1}{6} & \frac{2}{3} & \frac{1}{2} \\ 0 & -\frac{1}{6} & -\frac{1}{3} & \frac{1}{2} \end{bmatrix}$$

Figure 5 represents the transformed form of the K1 triangulation of a unit cube.

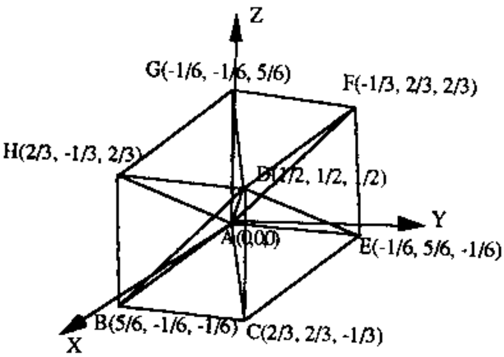

Figure 5. Transformed triangulated cube

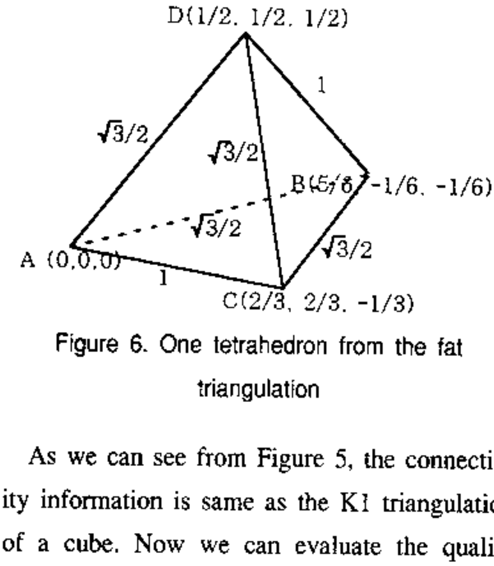

Figure 6. One tetrahedron from the fat triangulation

As we can see from Figure 5, the connectivity information is same as the K1 triangulation of a cube. Now we can evaluate the quality of the fat triangulation by examining the tetrahedron in Figure 6. One way of measuring the closeness of a tetrahedron to an equilateral tetrahedron is using a measure called normalized ratio. To calculate the normalized ratio, r/R (r: radius of the inscribed sphere for a given tetrahedron, R: radius of the circumscribed sphere for a given tetrahedron) is calculated first and this value is normalized to have the value of one for an equilateral tetrahedron. The normalized ratio for the transformed tetrahedron is calculated as follows:

$$9 \times \frac{\text{Volume}}{\text{SurfceArea}} \times \frac{1}{\text{CircumRadius}} \left(= 3 \times \frac{r}{R} \right)$$

$$= 9 \times \frac{\frac{1}{12}}{4 \times \frac{1}{2}\sqrt{2}} \times \frac{1}{\sqrt{5}}$$

$$= 3\frac{\sqrt{2}}{2}\sqrt{5}$$

$$= 0.9487$$

To compare the quality of the fat triangulation with other existing triangulation algorithms, we introduce a 3-D mesh generation algorithm which dealt with the quality of a mesh. [Field & Frey 85] In this algorithm, they used some properties of icosahedron. Icosahedron is a regular convex polyhedron which conists of 20 equilateral triangles on the surface. Figure 7 depicts an icosahedron.

To generate 3-D mesh, icosahedra are stacked up to fill up a 3-D space. For each icosahedron, a node is generated at the center of the icosahedron and vertices of the icosahedron is connected to the center node to generate tetrahedral elements inside the icosahedron. After stacking up icosahedra, there are empty spaces among the neighboring icosahedra. These spaces are also can be filled with tetrahedral elements and the normalized

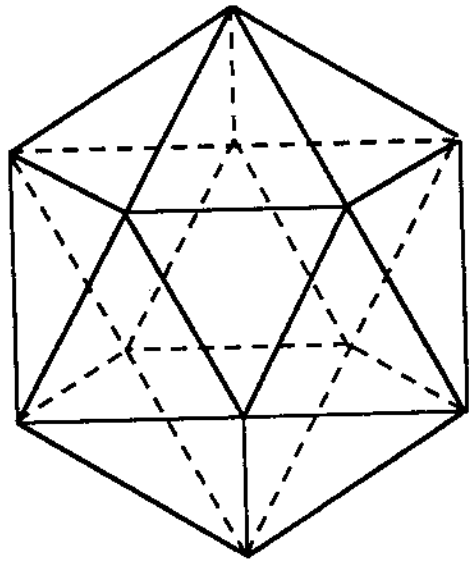

Figure 7. Icosahedron

ratio for these element could be calculated. The normalized ratio for the tetrahedral elements inside the icosahedron is 0.9964 and the smallest normalized ratio of the elements which fill the space among icosahedra is 0.7681. In another triangularion, where a cube is decomposed into 6 tetrahedra, the maximum and minimum values of normalized ratio are 0.73205 and 0.62299 each. Since the error of an analysis depends on the worst shaped element [Strang 76], fat triangulation is more suitable than any other approach known thus far in the finite element modeling. These results can be summarized as Table 1.

Table 1. Comparison of Triangulation

	Fat triangulation	Icosahedral triangulation	Cube trianguation
Maximum normalized ratio	0.9487	0.9964	0.73205
Minimum normalized ratio	0.9497	0.7681	0.62299

4. Application

The fat triangulation could be used in several ways to generate a 3-D mesh. For example, in the Delaunay triangulation approach [Cavendish 85], the shape of the element depends on the node distribution. Delaunay triangulation approach generates interior and boundary nodes inside and on a 3-D object and triangulates the object using a particular triangulation called Delaunay triangulation. One of the difficulties in this method is to generate interior node which introduces tetraheral elements with high normalized values. Topological and geometrical informations studied in this paper can be used to provide the interior node distribution.

In the other approach called modified octree approach, Shephard and Yerry in 1984 developed an algorithm generating tetrahedral elements using the idea of octree encoding. In this method, a cube is decomposed into 5 or 6 tetrahedra to make tetrahedral elements. We can apply the idea of fat triangulation in the octree-based approach. One modification is as follows. Apply the modified-octree approach to a parallelepiped which is the linear transformation done by multiplying K1 triangulation and A matrix from Equation 1. In this modification, a parallelepiped can be decomposed into 8 identical parallelepiped and each parallelepiped can be decomposed into 6 tetrahedra using the fat triangulation. This will provide the better shape of tetrahedra. Everything including graded mesh in the existing modified-octree approach can be directly used except the substitution of a cube with a parallelepiped. However, we can not use the integer tree structure. Although it is conjectured that a parallelepiped could be used instead of a cube in octree data structure without having any difficulty, possible break-downs should be closely reviewed.

5. Conclusion and Future Research

In this research, we investigated the properties of the Todd's fat triangulation and the application of the triangulation into the finite element mesh generation. In the topological aspect, the number of edges to be connected to an interior edge in order to be an equilateral uniform mesh as possible is identified as 14. This information can be used in the postprocessing step of a 3-D mesh generation algorithm where the connectivity of a mesh could be changed to improve the shape of the triangulation. In the geometrical aspect, large normalized ratio (0.9487) is identified which is the substantial improvements compare to the triangulations which are used in the existing 3-D mesh generation algorithms. Application of the fat triangulation into the Delaunay triangulation approach, and modified-octree approach were also illustrated. To complete the 3-D meshing algorithm, in both approaches, boundary node generation scheme and the triangulation scheme to connect boundary nodes and interior nodes should be investigated further. These remain as future research areas.

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