Journal of the Korean Institute
of Industrial Engineers
Vol.23, N0.4, December, 1997

Development of a New Methodology to find the Expected Busy Periods for Controllable M/G/1 Queueing Models Operating under the Multi-variable Operating Policies*: Concepts and applications to the dyadic policies

Hahn-Kyou Rhee**

Abstract

In this paper, steady-state controllable M/G/1 queueing systems operating under the dyadic policies are considered. A new method to obtain the expected busy period when the D-policy is involved in system operation, is developed. This new method requires derivation of so called 'the pseudo probability density function' of the busy period for the system under consideration, which is completely different from its actual probability density function. However, the proposed pseudo probability density function does generate the correct expected busy period through simple procedures.

Key words : D-policy, dyadic policy, busy period, pseudo assumption

1. Introduction

Consider a steady-state M/G/1 queueing system. Arrivals of customers to the system follow a Poisson process with intensity λ. Arriving customers form a single waiting line at the service station based on order of their arrivals. The total number of potential customers and the system capacity are assumed to be infinite. Service times for individual customers are identically and independently distributed, and dictated by a known arbitrary probability distribution with the cumulative distribution function $G(\cdot)$ and the probability density function $g(\cdot)$ with a finite mean $1/\mu$.

M/G/1 queueing models can be classified into two categories such as the ordinary and the controllable models based on availability of the server. In the ordinary models, the server is assumed to be always available for arriving customers. However, different from the ordinary models, the server could be removed from the system whenever there are no customers in the system to utilize

* This research was supported financially by Korea Science and Engineering Foundation in 1996 (GRANT NO. KOSEF 961-1012-085-1).

** Department of Industrial Engineering, Hannam University, Taejon, Korea

the server's idle time in the controllable models. Suppose that the server is removed from the system. Depending on when the removed server returns to the system to provide service, the following three simple operating policies have been introduced:

(i) The N-policy by Yadin and Naor[9]: If the number of customers in the system reaches N where ($N \geq 1$) for the first time after the server is removed from the system, the server returns immediately and provides service until there are no customers in the system again.

(ii) The T-policy by Heyman[5] or Levy and Yechiali [7]: The removed server returns to the system and provides service immediately after T time units have elapsed from the epoch of server removal if there is at least one customer present in the waiting line, until there are no customers in the system. If after T time units have elapsed there are no customers to initiate service, the server waits another T time units and so on, until at least one customer is shown in the waiting line.

(iii) The D-policy by Balachandran and Tijms[1]: If the server's workload or backlog which is equal to the sum of the service times of waiting customers exceeds D where ($D > 0$) for the first time, the removed server returns to the system and provides service to all customers until the system is empty.

These simple policies allow the control of an M/G/1 queueing system through judicious selection of the single decision variable N or T or D, by typically minimizing or maximizing a given objective function. Although, this type of control may be well suited for certain class of M/G/1 problems, nevertheless, other problems, identified by more complex objective function, may be better approached by introducing policies involving two decision variables. In fact, very often, it may be shown that the one-variable policy (simple policy) problems are nothing but special cases of the two-variable policy (dyadic policy) problems.

In the area of the controllable queueing models, the busy period is defined as the length of time from the epoch the server returns to the system to provide service to the epoch the server is removed from the system due to the fact that no customers are present in the system. The idle period is defined as the time elapsed between the end of a busy period and the beginning of the next busy period.

To obtain the expected busy period for controllable queueing models with dyadic policies, several approaches could be used. Among them, the following two methods are generally in use: (i) relationship between the probability that the server is busy and the expected busy and idle periods, and (ii) the probability density function of the busy period. Main idea of the first method is that the probability that the server is busy for any steady-state queueing models, denoted by $P[O]$, can be represented in terms of the mean arrival rate λ, the mean service rate μ, the expected busy period $E[B]$ and the expected idle period $E[I]$ such that

$$P[O] = \frac{E[B]}{E[B]+E[I]} = \frac{\lambda}{\mu}$$

Note that λ and μ denote the average numbers of customers' arrivals and service completions per unit time, respectively. As long as, the expected idle period is known or easy to derive, this method is strongly recommended. However, if not, this method may not be desirable. In other words, computational efforts and difficulties to derive the expected idle period is greater than those of the expected busy period, this method is not worth to consider. The second method could be considered as a formal approach to derive the expected busy period. Major problem involved in this method is, in general, derivation of the probability density function of the busy period for a queueing model requires a huge amount of computational efforts (Also see Rhee and Sivazlian[8] or Gakis, Rhee and Sivazlian[4].) In particular, if the operating policy for a controllable queueing model involves the D-policy, difficulties and complexities to derive it are tremendously increased because of definition of the decision variable D. That is, the server's initial workload which exceeds D and the number of waiting customers in the system when the server completes the initial workload, are important information to characterize the busy period. Note that the number of customers in the system when a busy period is initiated is sufficient information to characterize the busy period when either the N-policy or the T-policy is considered.

Main objective of this research is to develop a simple method to derive the expected busy periods for controllable M/G/1 queueing models operating under the dyadic policies which include the simple D-policy. Each of these policies is a different combination of the N-policy or the T-policy and the D-policy. Each policy has a peculiar structure and will be defined in the sequel. They are (i) the $Min(N,D)$ policy, (ii) the $Min(T,D)$ policy, (iii) the $Max(N,D)$ policy and (iv) the $Max(T,D)$ policy. Key idea of a new method for these policies is to utilize the number of customers in the system when a busy period is initiated similar to the cases for the N-policy and the T-policy, rather than using the server's initial workload which is greater than D and the number of customers in the system when the server completes the initial workload. Since the probability density functions and the expected values of the busy periods for these policies except the $Max(N,D)$ policy, are already obtained by Gakis et al.[4] using the probability density functions of the busy periods, simple procedures to derive the correct desired results from the proposed new method could be easily evaluated. To do so, first, concepts of the new method will be demonstrated for the simple D-policy. Then, the new method will be applied to three dyadic policies such as the $Min(N,D)$ policy, the $Min(T,D)$ policy and the $Max(N,D)$ policy to check whether it generates the desired results correctly and simply or not. Finally, the expected busy period for the $Max(T,D)$ policy which is not derived yet will be obtained by using the new method.

2. Definitions and Notations.

For an M/G/1 queueing model with the mean arrival rate λ and the mean service rate μ, define the followings:

(i) B_o : the random variable representing the busy period for an ordinary system with the probability density function, $f_{B_o}(\cdot)$. Then, the expected value of B_o, denoted by $E[B_o]$, is given by (e.g. see Conolly[3] or Kleinrock [6]).

$$E[B_o] = \frac{1}{\mu - \lambda}. \quad (1)$$

(ii) S_i for i=1,2,.. : the random variable representing the ith customer's service time with the common probability density function and cumulative distribution function $g(\cdot)$ and $G(\cdot)$, respectively.

(iii)
$$G^{(j)}(D) = \int_0^D g^{*(j)}(x)\,dx \quad \text{for, j=1,2,3,...} \quad (2)$$

$$G^{(0)} = 1 \quad (3)$$

where $g^{*(j)}(x)$ denotes the j fold convolution of $g(x)$.

(iv) $N(t)$: the random variable representing the number of customers in the system at time t. Then, the following holds since customers' arrivals follow a Poisson distribution with parameter λ

$$P[N(t) = n] = H_n(t) - H_{n+1}(t) \quad \text{for } n=0,1,2,3.. \quad (4)$$

where

$$H_n(t) = P[N(t) \geq n] = \sum_{j=n}^{\infty} \frac{e^{-\lambda t}(\lambda t)^j}{j!}.$$

(v) $\bar{f}(s)$: Laplace transform of a function $f(\cdot)$, i.e.,

$$\bar{f}(s) = \int_0^{\infty} e^{-sx} f(x)dx$$

(v) B_N : the random variable representing the busy period when the simple N policy is considered. Then, the expected value of B_N, denoted by $E[B_N]$, is given by

$$E[B_N] = N\ E[B_o] \tag{5}$$

(vi) B_T : the random variable representing the busy period when the simple T policy is considered. Then, the expected value of B_T, denoted by $E[B_T]$, is given by

$$E[B_T] = \frac{\lambda T\ E[B_o]}{1 - e^{-\lambda T}} \tag{6}$$

(vii) B_D : the random variable representing the busy period when the simple D policy is considered. Then, the expected value of B_D, denoted by $E[B_D]$, is given by

$$E[B_D] = E[B_o][M(D) + 1] \tag{7}$$

Note that $M(D)$ represents a renewal function in D time units, that is, the expected number of renewals or arrivals in D time units(e.g., see Bhat[2]) which satisfies

$$M(D) = \sum_{j=1}^{\infty} \int_0^D g^{*(j)}(x)dx$$
$$= \sum_{j=1}^{\infty} G^{(j)}(D) \tag{8}$$

$E[B_o]$ in (7) is the expected busy period for the ordinary system and shown in (1). For more information related to B_N, B_T and B_D, see Gakis et al.[4] or others.

3. A New Method to Derive the Expected Busy Period for the D-policy

As stated before, when the D-policy is considered, complexities and difficulties are encountered to characterize the busy period due to the definition of D. Different from the cases for the N-policy or T-policy, the number of customers in the system when a busy period is initiated, may not play important roles to characterize the busy period for the D-policy because the server's initial workload should be a part of the busy period. Main idea to develop a new method to derive the expected busy period for the D-policy is to utilize the number of customers in the system when a busy period is initiated, similar to the cases when the N-policy and the T-policy are considered. Hence, the server's initial workload itself is not a critical factor to characterize the busy period in the new method even though the D-policy is considered. More precisely, the following, so called, '*pseudo assumption*' is made:

Pseudo Assumption: *For the controllable M/G/1 queueing system with the D-policy, each of arriving customers before a busy period is initiated creates an identical busy period to the ordinary M/G/1 queueing system by ignoring the server's initial workload which exceeds D and the number of customers in the system when the server completes the initial workload. Hence, even though the D-policy is applied, it is possible to characterize the busy period based on the number of customers in the system when a busy period is initiated, similar to the cases for the N-policy and the T-policy.*

Then, consequently, this assumption leads to $0 \leq B_D < \infty$ rather than $D \leq B_D < \infty$. It is clear that the above pseudo assumption contradicts definition of the D-policy. Define $ff_{B_D}(\cdot)$ over $0 \leq B_D < \infty$ as the probability density function of the busy period for the D-policy under the pseudo assumption. Since $ff_{B_D}(\cdot)$ defined over $0 \le B_D < \infty$ is different from the actual probability density function of the busy period for the D-policy defined over $D \le B_D < \infty$, from now on, $ff_{B_D}(\cdot)$ is called the pseudo probability density function of the busy period. For characterizations of the $ff_{B_D}(\cdot)$, define the random variable J as minimum of j such that $\{S_1+S_2+...+S_j \ge D\}$ for $j \ge 1$ where S_j denotes the jth customer's service time. Then, the conditional pseudo probability density function of the busy period given that $J=j$, denoted by $ff_{B_D}(t|J=j)$, is derived from

$$ff_{B_D}(t|J=j) = f_{B_o}^{*(j)}(t) \quad \text{for} \quad j=1,2,3,.. \tag{9}$$

because j customers in the waiting line initiate a busy period. Thus, $ff_{B_D}(x)$ is obtained from

$$ff_{B_D}(t) = \sum_{j=1}^{\infty} ff_{B_D}(t|J=j)\, P[J=j] \tag{10}$$

However,

$$P[J=j]=P[S_1 \ge D], \quad \text{for} \quad j=1, \tag{11}$$

$$P[J=j]=P[S_1+S_2+...+S_j \ge D,\, S_1+S_2+...+S_{j-1} \le D], \quad \text{for} \quad j=2,3,4,.., \tag{12}$$

From (2), (11) and (12), it is clear that

$$P[J=1]=1 - G^{(1)}(D) \tag{13}$$

$$P[J=j]=G^{(j-1)}(D) - G^{(j)}(D) \quad \text{for} \quad j=2,3,4,.. \tag{14}$$

Recall the relation in (3), that is

$$G^{(0)}(D) = 1 \tag{15}$$

Thus, relations in (13), (14) and (15) together yield $P[J=j]$ for $j=1,2,3,...$ as

$$P[J=j]=G^{(j-1)}(D) - G^{(j)}(D) \tag{16}$$

Next, substituting $ff_{B_D}(x|J=j)$ in (9) and $P[J=j]$ in (16) into relation in (10) results the pseudo probability density function of the busy period, $ff_{B_D}(x)$ for $0 \le x < \infty$ as

$$ff_{B_D}(t) = \sum_{j=1}^{\infty} f_{B_o}^{*(j)}(t)\,[G^{(j-1)}(D) - G^{(j)}(D)] \tag{17}$$

Let $\overline{ff_{B_D}}(s)$ be Laplace transform of $ff_{B_D}(t)$. Then, it can be shown easily that

$$\overline{ff_{B_D}}(s) = \sum_{j=1}^{\infty} [\overline{f_{B_o}}(s)]^j\,[G^{(j-1)}(D) - G^{(j)}(D)] \tag{18}$$

Since $ff_{B_D}(t)$ in (17) is a proper probability density function due to the fact that setting s to zero in (18) yields one, differentiating relation in (18) with respect to s and setting s to zero yield the expected busy period for the D-policy under the pseudo assumption, denoted by $E[B_D]_p$, as

$$E[B_D]_p = E[B_o] \sum_{j=0}^{\infty} G^{(j)}(D) \tag{19}$$

From properties in a renewal process, the renewal function in time D units, $M(D)$ can be expressed in terms of interarrival time distributions which is equivalent to the customers' service times in this research as shown in (8). Thus, using relation in (3) and $M(D)$ in (8) together yields

$$M(D) + 1 = \sum_{j=0}^{\infty} G^{(j)}(D) \tag{20}$$

Therefore, from relations in (19) and (20), $E[B_D]_p$ can be rewritten as

$$E[B_D]_p = E[B_o]\,[M(D) + 1] \tag{21}$$

From relations in (7) and (21), it is clear that

$$E[B_D] = E[B_D]_p \tag{22}$$

The relation in (22) implies that the expected busy periods derived from the existing methods and the proposed method which uses the pseudo probability density functions of the busy period for the D-policy are identical.

4. Applications to the Dyadic Policies

Concepts of the pseudo assumption are now to be applied to four dyadic policies stated before such as (i) the $Min(N,D)$ policy, (ii) the $Min(T,D)$ policy, (iii) the $Max(N,D)$ policy and (iv) the $Max(T,D)$ policy, to derive the expected busy periods. First, the expected busy periods for the $Min(N,D)$ policy, the $Min(T,D)$ policy and the $Max(N,D)$ policy are derived by using the pseudo probability density functions. Since the desired results are already obtained by Gakis et al.[4] using the probability density functions of the busy periods, the new method will be easily checked whether it generates the desired results correctly through simple procedures or not. Then, it is applied to the $Max(T,D)$ policy to obtain the expected busy period which has not been derived yet.

4.1 The Min(N,D) policy

In the $Min(N,D)$ policy, following the start of an idle period, the server restarts serving and hence initiates a busy period, if either N customers have appeared in the system ($N \geq 1$) or the total accumulated backlog of customers' service times exceeds D, whichever occurs first. In other words, a busy period is initiated when the N-th customer arrives in the system if $S_1 + S_2 + \ldots + S_N < D$ or when the n-th customer arrives in the system if $S_1 + S_2 + \ldots + S_n > D$ ($n \leq N, n \geq 1$) for the first time. Note that when $N \to \infty$, the $Min(N,D)$ policy is equivalent to the simple D-policy, and when $D \to \infty$, the $Min(N,D)$ policy is equivalent to the simple N-policy. Also note that N=1, the $Min(N,D)$ policy is equivalent to the ordinary $M/G/1$ queueing system which is independent of D. From the result of Gakis et al.[4], the expected busy period for the $Min(N,D)$policy, denoted by $E[B_{Min(N,D)}]$, is given by

$$E[B_{Min(N,D)}] = E[B_o] \sum_{n=0}^{N-1} G^{(n)}(D) \tag{23}$$

However, their procedures to derive $E[B_{Min(N,D)}]$ from the probability density function of the busy period requires tremendous amount of computational efforts.

Let B_1 be the random variable representing the busy period when the $Min(N,D)$ policy is considered. Also let $ff_{B_1}(x)$ be the pseudo probability density function of B_1. Based on the definition of the $Min(N,D)$ policy and the number of customers in the system when a busy period is initiated, $ff_{B_1}(x)$ defined over $0 < B_1 < \infty$, is given by

$$ff_{B_1}(t) = \sum_{n=1}^{N-1} f_{B_o}^{*(n)}(t)[G^{(n-1)}(D) - G^{(n)}(D)] + f_{B_o}^{*(N)}(t)\, G^{(N)}(D) \tag{24}$$

Hence, Laplace transform of $ff_{B_1}(t)$ in (24), denoted by $\overline{ff_{B_1}}(s)$, is immediately given by

$$\overline{ff_{B_1}}(s) = \sum_{n=1}^{N-1} [\overline{f_{B_o}}(s)]^n [G^{(n-1)}(D) - G^{(n)}(D)] + [\overline{f_{B_o}}(s)]^N G^{(N)}(D) \tag{25}$$

Define $E[B_{Min(N,D)}]_p$ as the expected value of B_1 derived from $ff_{B_1}(x)$. Differentiating $\overline{ff_{B_1}}(s)$ in (25) with respect to s and setting s to zero yield $E[B_{Min(N,D)}]_p$ as

$$E[B_{Min(N,D)}]_p = E[B_O] \sum_{n=1}^{N} G^{(n-1)}(D) \tag{26}$$

From (23) and (26), it is clear that $E[B_{Min(N,D)}]$ and $E[B_{Min(N,D)}]_p$ are identical.

As stated before, $E[B_N]$ in (5) and $E[B_D]$ in (7) are recovered from either of $E[B_{Min(N,D)}]$ in (23) or $E[B_{Min(N,D)}]_p$ in (26) by tending $N \to \infty$ and $D \to \infty$, respectively.

4.2 The Min(T,D) policy

In the $Min(T,D)$ policy, following the start of an idle period, the server restarts serving and hence initiates a busy period, if either T time units have elapsed since the end of a busy period or the end of previous T time units and at least one customer has appearing during that time interval or the total accumulated backlog of customers' service time exceeds D, whichever comes first. Suppose that no customers appear in the first mT time units where m=0,1,2,... from the beginning of an idle period, and at least one customer appears in the next T time units. Then, the busy period is initiated when$(m+1)T$ time units have elapsed from the beginning of the idle period if the sum of customers' service times arriving during the $(m+1)$-st T time interval does not exceed D. The busy period is also initiated at the epoch when the n-th customer arrives in the system before$(m+1)T$ time units have elapsed if the sum of n customers' service times exceeds D for the first time. Similar to the previous $Min(N,D)$ policy, the D-policy and the T-policy are recovered from the $Min(T,D)$ policy by tending $T\to\infty$ and $D\to\infty$, respectively. Gakis et al.[4] derived the expected busy period for the $Min(T,D)$ policy, denoted by $E[B_{Min(T,D)}]$, from the probability density function of the busy period with large amount of computational efforts, as

$$E[B_{Min(T,D)}]=\frac{E[B_o]}{1-e^{-\lambda T}}\sum_{n=1}^{\infty}[H_n(T)-G^{(n-1)}(D)] \qquad (27)$$

Let B_2 and $ff_{B_2}(t)$ be the random variable representing the busy period for the $Min(T,D)$ policy and the pseudo probability density function of B_2, respectively. Then, based on the definition of the $Min(T,D)$ policy, relations in (4) and (16) and the number of customers in the system when a busy period is initiated, $ff_{B_2}(t)$ is obtained as

$$ff_{B_2}(t)=\sum_{m=0}^{\infty}e^{-m\lambda t}\{\sum_{n=1}^{\infty}f_{B_o}{}^{*(n)}(t)[H_n(T)-H_{n+1}(T)]G^{(n)}(D)$$
$$+\sum_{n=1}^{\infty}f_{B_o}{}^{*(n)}(t)H_n(T)[G^{(n-1)}(D)-G^{(n)}(D)]\} \qquad (28)$$

Let $\overline{ff_{B_2}}(s)$ be Laplace transform of $ff_{B_2}(t)$ in (28). Then,

$$\overline{ff_{B_2}}(s)=\frac{1}{1-e^{-\lambda T}}\sum_{n=1}^{\infty}[\overline{f_{B_o}}(s)]^n[H_n(T)G^{(n-1)}(D)-$$
$$H_{n+1}(T)G^{(n)}(D)] \qquad (29)$$

Let $E[B_{Min(T,D)}]_p$ be the expected value of B_2 derived from $ff_{B_2}(t)$. Differentiating $\overline{ff_{B_2}}(s)$ in (29) with respect to s and setting s to zero yield $E[B_{Min(T,D)}]_p$ as

$$E[B_{Min(T,D)}]_p=\frac{E[B_o]}{1-e^{-\lambda T}}\sum_{n=1}^{\infty}[H_n(T)-G^{(n-1)}(D)] \qquad (30)$$

which is identical to $E[B_{Min(T,D)}]$ in (27).

Also note that $E[B_T]$ in (6) and $E[B_D]$ in (7) are recovered from either of $E[B_{Min(N,D)}]$ in (27) or $E[B_{Min(N,D)}]_p$ in (30) by tending $D\to\infty$ and $T\to\infty$, respectively.

4.3 Max(N,D) policy

In the $Max(N,D)$ policy, following the start of an idle period, the server starts serving and hence initiates a busy period when at least N customers are present in the system and at least D units of total backlog of customers' service time have accumulated for the first time. Following an idle period it is possible that either with the arrival of the N-th customer, D units of total backlog have not accumulated and the server has to wait for more customers to arrive, or that while D units of total backlog have accumulated, the number of customer is less than N and the server has to wait until N customers are present. Of course a third possibility is that N-th customer is the one that exceeds that desired level of D units of accumulated total backlog. Clearly, any busy periods will exceed D time units. Note that the N-policy and the D-policy could be recovered from the $Max(N,D)$ policy by tending $D\to 0$, and $N\to 1$ respectively. Let B_3 be the random variable representing the busy period for the $Max(N,D)$ policy and $E[B_{Max(N,D)}]$ be the expected value of B_3. Then, from the results obtained by Gakis et al. [4], $E[B_{Max(N,D)}]$ is given by

$$E[B_{Max(N,D)}] = E[B_o]\sum_{n=N}^{\infty} G^{(n)}(D) + N\,E[B_o] \qquad (31)$$

Based on the fact that a busy period is initiated when the N-th customer arrives in the system if $S_1 + S_2 + \ldots + S_n \rangle D$ and $n \langle N$, or when the n-th customer arrives in the system if $S_1 + S_2 + \ldots + S_n \rangle D$ and $n \geq N$ for the first time in the $Max(N,D)$ policy, define the following probabilities:

(i) $P[N_3]$: the probability that a busy period is initiated when the N-th customer arrives in the system,

(ii) $P[D_3]$: the probability that a busy period is initiated when the sum of customers' service times exceeds D time units for the first time.

Then, $P[N_3]$ and $P[D_3]$ are given below:

$$P[N_3] = \sum_{n=1}^{N} [G^{(n-1)}(D) - G^{(n)}(D)] - [G^{(N-1)}(D) - G^{(N)}(D)]$$

$$= G^{(0)}(D) - G^{(N-1)}(D) \qquad (32)$$

$$P[D_3] = \sum_{n=N}^{\infty} [G^{(n-1)}(D) - G^{(n)}(D)] = G^{(N-1)}(D) \qquad (33)$$

Let $ff_{B_3}(t)$ be the pseudo probability density function of B_3. Then, using the definition of the $Max(N,D)$ policy, $P[N_3]$ in (32), $P[D_3]$ in (33) and the number of customers in the system when a busy period is initiated, yields $ff_{B_3}(t)$ as

$$ff_{B_3}(t) = f_{B_o}^{*(N)}(t)[G^{(0)}(D) - G^{(N-1)}(D)]$$

$$+ \sum_{n=N}^{\infty} f_{B_o}^{*(n)}(t)[G^{(n-1)}(D) - G^{(n)}(D)] \qquad (34)$$

Thus, Laplace transform of $ff_{B_3}(t)$ in (34), denoted by $\overline{ff_{B_3}}(s)$, is obtained as

$$\overline{ff_{B_3}}(s) = [\overline{f_{B_o}}(s)]^N[G^{(0)}(D) - G^{(N-1)}(D)]$$

$$+ \sum_{n=N}^{\infty} [\overline{f_{B_o}}(s)]^n[G^{(n-1)}(D) - G^{(n)}(D)] \qquad (35)$$

Let $E[B_{Max(N,D)}]_p$ be the expected value of B_3 derived from $ff_{B_3}(t)$. Differentiating $\overline{ff_{B_3}}(s)$ in (35) with respect to s and then setting s to zero yield $E[B_{Max(N,D)}]_p$ as

$$E[B_{Max(N,D)}]_p = E[B_O][N + \sum_{n=N}^{\infty} G^{(n)}(D)] \qquad (36)$$

Clearly, $E[B_{Max(N,D)}]$ in (31) and $E[B_{Max(N,D)}]_p$ in (36) are identical. Note that $E[B_N]$ in (5) and $E[B_D]$ in (7) are recovered from either of $E[B_{Max(N,D)}]$ in (31) or $E[B_{Max(N,D)}]_p$ in (36) by tending $D \to 0$ and $N \to 1$, respectively.

Based on the known results obtained from existing methods and the results obtained from the pseudo probability density functions of the busy periods for the $Min(N,D)$ policy, the $Min(T,D)$ policy and the $Max(N,D)$ policy, it is clear that the new method does generate the correct expected busy periods with simple procedures. Finally, the pseudo probability density function is now applied to the $Max(T,D)$ policy to obtain the expected busy period which is not derived yet.

4.4 The Max(T,D) policy

Suppose that no customers arrive in the system during the first $mT(m=0,1,2,\ldots)$ time units from the end of the previous busy period and $n(n \geq 1)$ customers arrive in the system during the next T time units. In the $Max(T,D)$ policy, following the start of an idle period, the server starts serving and hence initiates a busy period either when $(m+1)T$ time units have elapsed from the end of previous busy period if $S_1 + S_2 + \ldots + S_n \geq D$, or when $S_1 + S_2 + \ldots + S_k \geq D$ where $k \rangle n$ is satisfied for the first time after $(m+1)T$ time units have elapsed from the end of previous busy period. Thus, any busy periods can not be initiated before $(m+1)T$ time units have elapsed from the start of an idle period. Note that tending $T \to 0$ and $D \to 0$ in the $Max(T,D)$ policy yield the simple D-policy and T-policy, respectively. Define the followings:

(i) $P[T_4]$: the probability that a busy period is initiated when $(m+1)T$ time units have elapsed from the end of previous busy period,

(ii) $P[D_4]$: the probability that a busy period is initiated when accumulated backlog for the customers' service times exceeds D for the first time after $(m+1)T$ time units have elapsed from the end of previous busy period,

(iii) B_4 : the random variable representing the busy period for the $Max(T,D)$ policy,

(iv) $ff_{B_3}(t)$: the pseudo probability density function of B_4,

(v) $\overline{ff_{B_4}}(s)$: Laplace transform of $ff_{B_4}(t)$,

(v) $E[B_{Max(T,D)}]_p$: the expected value of B_4 derived from $ff_{B_4}(t)$.

Then, based on the definition of the $Max(T,D)$ policy, the followings hold:

$$P[T_4] = \sum_{n=1}^{\infty}[H_n(T)-H_{n+1}(T)]\sum_{j=1}^{n}[G^{(j-1)}(D)-G^{(j)}(D)]$$
$$= \sum_{n=1}^{\infty}[H_n(T)-H_{n+1}(T)][G^{(0)}(D)-G^{(n)}(D)] \quad (37)$$

$$P[D_4] = \sum_{n=1}^{\infty}[H_n(T)-H_{n+1}(T)]\sum_{j=n+1}^{\infty}[G^{(j-1)}(D)-G^{(j)}(D)] \quad (38)$$

Using $P[T_4]$ in (37) and $P[D_4]$ in (38) and the number of customers in the system when a busy period is initiated together yields $ff_{B_4}(t)$ as

$$ff_{B_4}(t) = \sum_{m=0}^{\infty} e^{-m\lambda T}\{\sum_{n=1}^{\infty} f_{B_o}^{*(n)}(t)[H_n(T)-H_{n+1}(T)] \cdot [G^{(0)}(D)-G^{(n)}(D)] + \sum_{n=1}^{\infty}[H_n(T)-H_{n+1}(T)] \sum_{j=n+1}^{\infty} f_{B_o}^{*(j)}(t)[G^{(j-1)}(D)-G^{(j)}(D)]\}$$
$$= \frac{1}{1-e^{-\lambda T}}\{\sum_{n=1}^{\infty} f_{B_o}^{*(n)}(t)[H_n(T)-H_{n+1}(T)] \cdot [G^{(0)}(D)-G^{(n)}(D)] + \sum_{n=1}^{\infty}[H_n(T)-H_{n+1}(T)] \sum_{j=1}^{\infty} f_{B_o}^{*(n+j)}(t)[G^{(n+j-1)}(D)-G^{(n+j)}(D)]\} \quad (39)$$

Thus, Laplace transform of $ff_{B_4}(t)$ in (39), denoted by $\overline{ff_{B_4}}(s)$ is obtained as

$$\overline{ff_{B_4}}(s) = \frac{1}{1-e^{-\lambda T}}\{\sum_{n=1}^{\infty} [\overline{f_{B_o}}(s)]^n[H_n(T)-H_{n+1}(T)] \cdot [G^{(0)}(D)-G^{(n)}(D)] + \sum_{n=1}^{\infty}[H_n(T)-H_{n+1}(T)] \cdot \sum_{j=1}^{\infty} [\overline{f_{B_o}}(s)]^{n+j}\{G^{(n+j-1)}(D)-G^{(n+j)}(D)]\} \quad (40)$$

Differentiating $\overline{ff_{B_4}}(s)$ in (40) with respect to s and tending s to zero yield $E[B_{Max(T,D)}]_p$ as

$$E[B_{Max(T,D)}]_p = \frac{E[B_o]}{1-e^{-\lambda T}}\sum_{n=1}^{\infty}\{H_n(T) + [H_1(T)-H_{n+1}(T)]G^{(n)}(D)\}$$
$$= \frac{E[B_o]}{1-e^{-\lambda T}}\sum_{n=1}^{\infty}\{\lambda T + [H_1(T)-H_{n+1}(T)]G^{(n)}(D)\} \quad (41)$$

From $E[B_{Max(T,D)}]_p$ in (41), it is easy to verify that

$$\lim_{D\to\infty} E[B_{Max(T,D)}]_p = E[B_T]$$

$$\lim_{T\to\infty} E[B_{Max(T,D)}]_p = E[B_D]$$

Note that $E[B_T]$ and $E[B_D]$ are in (6) and (7), respectively. Since $E[B_{Max(T,D)}]_p$ in (41) also satisfies the following relation, it can be stated that the pseudo probability density function also generates the correct expected busy period for the $Max(T,D)$ policy.

$$E[B_{Min(T,D)}] + E[B_{Min(T,D)}]_p = E[B_T] + E[B_D]$$

Also note that

$$E[B_{Min(N,D)}] + E[B_{Min(N,D)}] = E[B_N] + E[B_D]$$

5. Summary

In this paper, a new methodology to derive the expected busy period for controllable M/G/1 queueing systems involving D-policy has been developed. It has been shown that the new method generates the correct values for the simple D-policy and four dyadic policies which are the $Min(N,D)$ policy, the $Min(T,D)$ policy, the $Max(N,D)$ policy and the $Max(T,D)$ policy by using the pseudo probability density functions of the busy periods. Important contribution of this research is not derivations of the expected busy periods for such policies, but relatively simple procedures to derive the desired results. This implies that there exists an additional method to derive the expected busy periods when the D policy is involved in system operations. Furthermore, the concepts of the pseudo probability density function of the busy period provides possibilities to analyze more complex operating policies like the triadic policies which includes three decision variables. Such policies could be considered as more generalized and flexible operating policies than the ordinary system, the simple policies and the dyadic policies. Sometimes, it might be desirable to analyze certain complex problems from different point of view because there could be another ways to solve them without much difficulties. However, readers of this paper should be advised not to have the following misunderstandings: (i) The proposed method in this paper is not the simplest way to derive the expected busy period for the simple D-policy. (ii) The proposed pseudo probability density function of the busy period does not necessarily guarantee to generate correct higher moments of the busy period. (iii) The concepts of pseudo asslumption may not be applicable to derive other system characteristics such as the expected number of customers in the system and so on.

References

[1] Balachandran, K.R. and Tijms, H., "On the D-policy for the M/G/1 Queue", *Management Science*, Vol. 21, No. 9, pp1073-1076, 1975.

[2] Bhat, N., "Elements of Applied Stochastic Process", 2nd ed., John Wiley & Sons, Inc, New York, 1984.

[3] Conolly, B., "Lecture Notes on Queueing Systems", Halsted, New York, 1975.

[4] Gakis, K.G., Rhee, H.K. and Sivazlian, B.D., "Distributions and First Moments of the Busy and Idle Periods in Controllable M/G/1 Queueing Models with Simple and Dyadic Policies", *Stochastic Analysis and Applications*, Vol. 13, No. 1, pp47-81, January, 1995.

[5] Heyman, D., "The T-policy for the M/G/1 Queue", *Management Science*, Vol. 23, No. 7, pp775-778, 1977.

[6] Kleinrock, L., "Queueing System, Volume 1: Theory", John Wiley & Sons, Inc, New York, 1975.

[7] Levy, Y., and Yechiali, U., "Utilization of Idle Time in an M/G/1 Queueing System", *Management Science*, Vol. 22, No. 2, pp202-211, 1976.

[8] Rhee, H.K. and Sivazlian, B.D., "Distribution of the Busy Period in a Controllable M/M/2 Queue Operating under the Triadic (0,K,M,N) Policy", *Journal of Applied Probability*, Vol. 27, pp425-432, 1990.

[9] Yadin, M. and Naor, P., "Queueing System with Removable Service Station", *Operational Research Quarterly*, Vol. 14, pp393-405, 1963.

97년 4월 최초 접수, 97년 8월 최종 수정