Wavelets and Filter Banks

Inheung Chon

Department of Mathematics
Seoul Woman's University
Seoul 139-774, Korea

Abstract

We show that if an even length filter has the same length complementary filter in a generalized linear phase case, the complementary filter is unique, we find sufficient conditions for a unique existence of even length N complementary filter in a quadrature mirror filter bank, and we find all higher degree symmetric filters of length $N + 4m$ which are complementary to a given symmetric filter of even length N.

1 Introduction

Wavelet theory has attracted much attention from many researchers in mathematics and electrical engineering fields. The integral wavelet transform and wavelet series, which are main topics in wavelet theory, are performed using wavelets. Thus constructing proper wavelets for applications is considered to be very important. Filter banks are efficient convolution structures that have been used in subband coders for speech. In a filter bank, a data sequence $x(n)$ is decomposed into M channels by convolution with sequences $h_i(n)$, called the analysis filters, down-sampled by M on each channel, upsampled by M on each channel, and then convolved with the sequences $g_i(n)$, called the synthesis filters, and recombined to give $\hat{x}(n)$. An important problem in a filter bank theory is how to design perfect reconstruction filter banks.

1991 Mathematics Subject Classification. 94A12.

Key words and phrases. quadrature mirror filter bank, perfect reconstruction, valid polynomial, complementary filter.

This paper was supported (in part) by the research fund of Seoul Women's Unuversity.

Daubechies ([4]) constructed compactly supported orthonormal wavelets and orthonormal scaling functions based on iterations of discrete filters and Vetterli ([7]) showed that biorthogonal perfect reconstruction filter banks generate biorthogonal wavelets and biorthogonal scaling functions. In order to generate wavelets using perfect reconstruction filter banks, it is required to find high pass complementary filter with respect to a given lowpass filter.

In section 2, we review some basics of scaling function, wavelets, and filter banks and we prove that a Laurent series under some assumptions generates orthonormal scaling function ϕ and orthonormal wavelet ψ and generated ϕ and ψ lead to perfect reconstruction filter bank, which is a variant of Vetterli's work ([7]). In section 3, we show that if an even length filter has the same length complementary filter in a generalized linear phase case, the complementary filter is unique, we find sufficient conditions for a unique existence of even length N complementary filter in a quadrature mirror filter bank, and we find all higher degree symmetric filters of length $N + 4m$ which are complementary to a given symmetric filter of even length N.

2 Wavelets and Filter Banks

Definition 2.1. A function $\psi \in L^2(\mathbb{R})$ is called an $\mathcal{R} - function$ if $\{\psi_{j,k} := 2^{2/j}\psi(2^j x - k)\}$ is a Riesz basis of $L^2(\mathbb{R})$ in the sense that the linear span of $\psi_{j,k}$ is dense in $L^2(\mathbb{R})$ and positive constants A and B exist, with $0 < A \leq B < \infty$, such that

$$A \parallel \{c_{j,k}\} \parallel_{l^2}^2 \leq \parallel \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} c_{j,k}\psi_{j,k} \parallel_2^2 \leq B \parallel \{c_{j,k}\} \parallel_{l^2}^2$$

for all doubly bi-infinite square-summable sequences $\{c_{j,k}\}$; that is,

$$\parallel \{c_{j,k}\} \parallel_{l^2}^2 := \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} |c_{j,k}| < \infty.$$

Definition 2.2. A function $\phi \in L^2(\mathbb{R})$ is called a *scaling fuction* if the subspaces V_j of $L^2(\mathbb{R})$ defined by

$$V_j := \overline{\langle \phi_{j,k} : \ k \in \mathbb{Z} \rangle}, \quad j \in \mathbb{Z},$$

satisfy

(1) $\cdots \subset V_{-1} \subset V_0 \subset V_1 \subset \dots$

(2) $\bigcup_{j\in\mathbb{Z}} = L^2(\mathbb{R})$

(3) $f(x) \in V_j$ if and only if $f(2x) \in V_{j+1},\ j \in \mathbb{Z}$

(4) $f(x) \in V_j$ if and only if $f(x + \frac{1}{2^j}) \in V_j,\ j \in \mathbb{Z}$

and if $\{\phi(\cdot - k) :\ k \in \mathbb{Z}\}$ is a Riesz basis of V_0. We say that the scaling function ϕ generates a multiresolution analysis $\{V_j\}$ of $L^2(\mathbb{R})$. Hereafter multiresolution analysis is abbreviated by MRA.

It is well known (see [3]) that if $\phi \in L^2(\mathbb{R})$ is a scaling function that generates an MRA $\{V_j\}$ of $L^2(\mathbb{R})$, then $\cap_{j\in\mathbb{Z}} V_j = \{0\}$.

Definition 2.3. If a scaling function ϕ generates an MRA, then there exists a unique sequence $\{p_k\}$ such that $\phi(x) = \sum_{k=-\infty}^{\infty} p_k \phi(2x-k)$. We call the sequence $\{p_k\}$ *two scale sequence of* ϕ. Corresponding to this l^2 sequence, we define *two scale symbol* of ϕ by

$$P(z) = P_\phi(z) := \frac{1}{2} \sum_{k=-\infty}^{\infty} p_k z^k.$$

Definition 2.4. A function $\psi \in L^2(\mathbb{R})$ is called an *orthogonal wavelet* if the family

$$\{\psi_{j,k}(x) := 2^{j/2}\psi(2^j x - k) : j, k \in \mathbb{Z}\}$$

is an orthonormal basis of $L^2(\mathbb{R})$; that is,

$$\langle \psi_{j,k}, \psi_{l,m} \rangle = \delta_{j,l}\delta_{k,m}, \quad j, k, l, m \in \mathbb{Z},$$

and every $f \in L^2(\mathbb{R})$ can be written as

$$f(x) = \sum_{j,k=-\infty}^{\infty} c_{j,k}\psi_{j,k}(x),$$

where the convergence of the series is in $L^2(\mathbb{R})$.

Definition 2.5. A collection of filters with a common input or a common output is called a *filter bank*. The filter bank which splits a signal $x(n)$ into M signals $x_0(n), x_1(n), \dots, x_{M-1}(n)$ is called an *analysis bank* and the filter bank which collects M signals $y_0(n), y_1(n), \dots, y_{M-1}(n)$ into a signal $\hat{x}(n)$ is called a *synthesis* *bank*. The splitted signals $x_0(n), x_1(n), \ldots, x_{M-1}(n)$ are called *subband signals*.

Definition 2.6. The device which takes an input sequene $x(n)$ and produces the output sequence $y_D(n) = x(Mn)$ with a positive integer M is called an *M-fold decimator*, a *downsampler*, or a *subsampler*. The device which takes an input $x(n)$ and produces an output sequence

$$y_E(n) = \begin{cases} x(\frac{n}{L}), & n \text{ is integer multiple of positive integer } L \\ 0, & \text{otherwise}. \end{cases}$$

is called an *L-fold expander*, an *upsampler*, or a *intepolator*.

We review the following basic proposition which will be used for Theorem 3.4.

Proposition 2.7. *If a filter bank has $H_0(z)$ and $H_1(z)$ as its analysis filters with downsampling by two and has $F_0(z)$ and $F_1(z)$ as its synthesis filters with upsampling by two, then the output $\hat{X}(z)$ of the filter bank with its input $X(z)$ is given by*

$$\hat{X}(z) = \frac{X(z)[H_0(z)F_0(z) + H_1(z)F_1(z)] + X(-z)[H_0(-z)F_0(z) + H_1(-z)F_1(z)]}{2}.$$

Particularly, if $F_0(z) = H_1(-z)$ and $F_1(z) = -H_0(-z)$, the output $\hat{X}(z)$ is given by

$$\hat{X}(z) = \frac{X(z)[H_0(z)F_0(z) + H_1(z)F_1(z)]}{2} = \frac{X(z)[H_0(z)H_1(-z) - H_0(-z)H_1(z)]}{2}.$$

Proof. Let $a(n)$ be the input and $b(n)$ be the output of the upsampler by two. Then

$$B(z) = \sum_{n=-\infty}^{\infty} b(n)z^{-n} = \sum_{m=-\infty}^{\infty} b(2m)z^{-2m} = \sum_{m=-\infty}^{\infty} a(m)z^{-2m} = A(z^2).$$

Let $c(n)$ be the input and $d(n)$ be the output of the downsampler by two. Then

$$D(z) = \sum_{n=-\infty}^{\infty} d(n)z^{-n} = \sum_{m=-\infty}^{\infty} c(2n)z^{-n}.$$

$$\frac{1}{2}(C(z^{\frac{1}{2}} + C(-z^{\frac{1}{2}})) = \frac{1}{2}(\sum_{n=-\infty}^{\infty} c(n)z^{-\frac{n}{2}} + \sum_{n=-\infty}^{\infty} c(n)(-1)^{-n}z^{-\frac{n}{2}}) = \sum_{n=-\infty}^{\infty} c(2n)z^{-n}.$$

Thus

$$D(z) = \frac{1}{2}(C(z^{\frac{1}{2}}) + C(-z^{\frac{1}{2}})).$$

The output of $H_0(z)$ and downsampler is

$$\frac{1}{2}[H_0(z^{0.5})X(z^{0.5}) + H_0(-z^{0.5})X(-z^{0.5})].$$

The output of $H_1(z)$ and downsampler is

$$\frac{1}{2}[H_1(z^{0.5})X(z^{0.5}) + H_1(-z^{0.5})X(-z^{0.5})].$$

The output of upsampler and $F_0(z)$ is

$$\frac{1}{2}[H_0(z)X(z)F_0(z) + H_0(-z)X(-z)F_0(z)].$$

The output of upsampler and $F_1(z)$ is

$$\frac{1}{2}[H_1(z)X(z)F_1(z) + H_1(-z)X(-z)F_1(z)].$$

Thus the proposition is proved. ■

Definition 2.8. Suppose the filter bank has $X(z)$ as its input and has $\hat{X}(z)$ as its output. If $\hat{X}(z) = kz^{-d}X(z)$ for some constant k and $d \in \mathbb{Z}$, we say the filter bank has a *perfect reconstruction* –hereafter denoted by PR– property. A polynomial $P(z)$ which satisfies

$$P(z) - P(-z) = kz^{-2l-1} \text{ for some constatant } k \text{ and } l \in \mathbb{Z}$$

is called a *valid polynomial*. Given a filter $H_0(z)$, any filter $H_1(z)$ such that $P(z) = H_0(z)H_1(-z)$ is a valid polynomial is called a *complementary filter*. A Laurent series is said to belong to the *Wiener Class* $\mathcal{W}$ if its coefficients sequence is in l^1.

Theorem 2.9. *Let $H_0(z) = \sum_{n=0}^{L-1} h_0(n)z^{-n}$ and $H_1(z) = \sum_{n=0}^{L-1} h_1(n)z^{-n}$ be filters such that $\langle h_0(n-2k), h_0(n-2l)\rangle = \delta_{kl}$ and $h_1(n) = (-1)^n h_0(L-1-n)$. Then the filter bank which has $H_0(z)$ and $H_1(z)$ as its analysis filters with downsampling two and has $G_0(z) = \sum_{n=0}^{L-1} h_0(L-1-n)$ and $G_1(z) = \sum_{n=0}^{L-1} h_1(L-1-n)$ as its synthesis filters with upsampling two has a perfect reconstruction property.*

Proof. See [7]. ■

Daubechies and other researchers (see [4], [5], and [7]) characterized the relationships between wavelets and filter banks. The followng theorem is a variant of Vetterli's work (see [7]).

Theorem 2.10. *Let $P \in \mathcal{W}$ be a Laurent series such that*

$$P(z) = \frac{1}{2}\sum_k p_k z^k = (\frac{1+z}{2})^N S(z),$$

where N is some positive integer and $S \in \mathcal{W}$ satisfies $S(1) = 1$. Let $S(z) = \sum_k s_k z^k$ and $B := \max_{|z|=1} |S(z)|$. If $|P(z)|^2 + |P(-z)|^2 = 1$, $|z| = 1$, $\sum_k |s_k||k|^\alpha < \infty$ for some $\alpha > 0$, and $B < 2^{N-1}$, then the scaling function ϕ and wavelet ψ obtained from $P(z)$ generates perfect reconstruction finite impulse response filter bank.

Proof. Let $\phi(x) := \sum_{k\in\mathbb{Z}} p_k \phi(2x-k)$. Then ϕ is an orthonormal scaling function that generates an MRA of $L^2(\mathbb{R})$ (see [3]). Let $\psi(x) := \sum_{k\in\mathbb{Z}} (-1)^k p_{-k+1}\phi(2x-k)$. Then

$$\langle \phi_{j,k}, \phi_{j,l}\rangle = \delta_{k,l}, \ \ j,k,l \in \mathbb{Z},$$
$$\langle \phi_{j,k}, \psi_{j,l}\rangle = 0, \ \ j,k,l \in \mathbb{Z},$$
$$\langle \psi_{j,k}, \psi_{l,m}\rangle = \delta_{j,l}\delta_{k,m}, \ \ j,k,l,m \in \mathbb{Z}.$$

That is, ψ is an orthonormal wavelet. Hence

$$\langle \phi(2x-l), \phi(2x-k)\rangle = \int_{-\infty}^{\infty} \phi(2x-l)\phi(2x-k)dx = \frac{1}{2}\int_{-\infty}^{\infty} \phi(y-l)\phi(y-k)dy = \frac{1}{2}\delta_{k,l}.$$

$$\langle \phi(x+k), \phi(x+l)\rangle = \langle \sum_n p_n\phi(2x+2k-n), \sum_m p_m\phi(2x+2l-m)\rangle = \frac{1}{2}\sum_n p_{n+2k}p_{n+2l}.$$

Since $\langle \phi(x+k), \phi(x+l)\rangle = \delta_{kl}$, $\frac{1}{2}\sum_n p_{n+2k}p_{n+2l} = \delta_{kl}$. Let $H_0(z)$ be a finite length filter such that $h_0(n) = \frac{p_n}{\sqrt{2}}$. Then

$$\langle h_0(n+2k), h_0(n+2l)\rangle = \langle \frac{p_{n+2k}}{\sqrt{2}}, \frac{p_{n+2l}}{\sqrt{2}}\rangle = \frac{1}{2}\sum_n p_{n+2k}p_{n+2l} = \delta_{kl}.$$

Thus $H_0(z)$ is an orthogonal filter.

Let $H_1(z)$ be a finite length filter such that $h_1(n) = \frac{(-1)^n p_{L-1-n}}{\sqrt{2}}$. Then

$$\langle h_1(n+2k), h_1(n+2l)\rangle = \langle \frac{p_{L-1-n-2k}}{\sqrt{2}}, \frac{p_{L-1-n-2l}}{\sqrt{2}}\rangle = \frac{1}{2}\sum_n p_{L+1-n-2k}p_{L+1-n-2l} = \delta_{kl}.$$

Thus $H_1(z)$ is an orthogonal filter.

Let $G_0(z) = \sum_{n=0}^{L-1} h_0(L-1-n)$ and $G_1(z) = \sum_{n=0}^{L-1} h_1(L-1-n)$. By Theorem 2.9, the filter bank which has $H_0(z)$ and $H_1(z)$ as its analysis filters with downsampling two and has $G_0(z)$ and $G_1(z)$ as its synthesis filters with upsampling two has a perfect reconstruction property. ∎

3 Complementary Filters

In section 2, we prove that a Laurent series under some assumptions generates orthonormal scaling function ϕ and orthonormal wavelet ψ and generated ϕ and ψ lead to perfect reconstruction filter bank. In this section, we consider a generation of wavelets using perfect reconstruction filter banks as the reverse problem of section 2. Since it is required to find high pass complementary filter with respect to a given lowpass filter for a generation of wavelets, we find sufficient conditions for a unique existence of even length N complementary filter in a quadrature mirror filter bank and we find all higher degree symmetric filters of length $N+4m$ which are complementary to a given symmetric filter of even length N.

Definition 3.1. The filters $H_0(z)$ and $H_1(z)$ which satisfy $H_1(z) = H_0(-z)$, that is, $|H_1(e^{i\omega})| = |H_0(e^{i(\pi-\omega)})|$, are called *quadrature mirror filters* and are abbreviated by QMFs. That is, $|H_1(e^{i\omega})|$ is a mirror image of $|H_0(e^{i\omega})|$ with respect to the quadrature frequency $\frac{2\pi}{4}$. The filter bank which has $H_0(z)$ and $H_1(z) = H_0(-z)$ as its analysis filters with downsampling by two and $F_0(z) = H_0(z)$ and $F_1(z) = -H_1(z)$ as its synthesis filters with upsampling by two is called a *quadrature mirror filter bank* and is abbreviated by QMFB.

Theorem 3.2.

(1) A real valued sequence $\{a_n\} \in l^1$ has generalized linear phase iff it is symmetric or antisymmetric (with respect to the phase of the symbol of $\{a_n\}$).

(2) A real valued finite sequence $\{a_n\} \in l^1$ with support $[0, N]$ has linear phase iff $a_{N-n} = a_n$ and $A(z) = \sum_{n=0}^{N} a_n z^n$ has only zeros of even order on the unit circle.

Proof. See [3]. ■

Theorem 3.3. *In the generalized linear phase case, if a filter $H_0(z)$ of an even length N has a complementary filter $H_1(z)$ of length N, then $H_1(z)$ is unique.*

Proof. By Theorem 3.2, we consider the following four cases.
(i) $H_0(z)$ and $H_1(z)$ are symmetric
$P(z) = H_0(z)H_1(z) = [h_0(0)h_1(0)] + [h_0(1)h_1(0) - h_0(0)h_1(1)]z^{-1} + \cdots + [h_0(0)h_1(1) - h_0(1)h_1(0)]z^{-(2N-3)} - h_0(0)h_1(0)z^{-(2N-2)}$. Thus $P(z)$ is antisymmetric except the central term. $P(z) - P(-z) = 2[h_0(1)h_1(0) - h_0(0)h_1(1)]z^{-1} + \cdots + 2[h_1(1)h_0(0) - h_0(1)h_1(0)]z^{-(2N-3)} = 2z^{-2l-1}$. Hence $P(z) - P(-z)$ is antisymmetric except the central term. We have the following system of equations

where $h_0(0), h_0(1), \ldots h_0(N-1)$ are given and $h_1(0), h_1(1), \ldots h_1(N-1)$ are unknowns.

$$h_0(1)h_1(0) - h_0(0)h_1(1) = 0$$
$$h_0(2)h_1(0) + h_0(0)h_1(2) - h_0(1)h_1(1) = 0$$
$$\ldots$$
$$-h_0(2)h_1(0) - h_0(0)h_1(2) + h_0(1)h_1(1) = 0$$
$$h_1(1)h_0(0) - h_0(1)h_1(0) = 0$$

This system consists of N equations but actually consists of $\frac{N}{2}$ equations by antisymmetry. By symmetry of $H_1(z)$, the number of unknowns is $\frac{N}{2}$. Thus the system consisting of $\frac{N}{2}$ equations has $\frac{N}{2}$ unknowns. Hence the system has a unique solution if the solution exists.

(ii) $H_0(z)$ is symmetric and $H_1(z)$ is antisymmetric

$P(z) = H_0(z)H_1(-z) = [h_0(0)h_1(0)] + [h_0(1)h_1(0) - h_0(0)h_1(1)]z^{-1} + \cdots + [-h_0(0)h_1(1)$
$+h_0(1)h_1(0)]z^{-(2N-3)} + h_0(0)h_1(0)z^{-(2N-2)}$. Thus $P(z)$ is symmetric except the central term. $P(z) - P(-z) = 2[h_0(1)h_1(0) - h_0(0)h_1(1)]z^{-1} + \cdots + 2[-h_1(1)h_0(0) + h_0(1)h_1(0)]z^{-(2N-3)} = 2z^{-2l-1}$. Hence $P(z) - P(-z)$ is symmetric except the central term. We have the following system of equations where $h_0(0), h_0(1), \ldots h_0(N-1)$ are given and $h_1(0), h_1(1), \ldots h_1(N-1)$ are unknowns.

$$h_0(1)h_1(0) - h_0(0)h_1(1) = 0$$
$$h_0(2)h_1(0) + h_0(0)h_1(2) - h_0(1)h_1(1) = 0$$
$$\ldots$$
$$h_0(2)h_1(0) + h_0(0)h_1(2) - h_0(1)h_1(1) = 0$$
$$-h_1(1)h_0(0) + h_0(1)h_1(0) = 0$$

This system consists of N equations but actually consists of $\frac{N}{2}$ equations by symmetry. By antisymmetry of $H_1(z)$, the number of unknowns is $\frac{N}{2}$. Thus the system consisting of $\frac{N}{2}$ equations has $\frac{N}{2}$ unknowns. Hence the system has a unique solution if the solution exists.

(iii) $H_0(z)$ and $H_1(z)$ are antisymmetric

The proof is similar to that of the case (i)

(iv) $H_0(z)$ is antisymmetric and $H_1(z)$ is symmetric

The proof is similar to that of the case (ii) ■

Theorem 3.4. *If a symmetric filter $H_0(z)$ in QMFB has an even length N and* *the greatest common divisor of $H_0(z)$ and $H_0(-z)$ divides $2z^{-2l-1}$ for some $l \in \mathbb{Z}$, then $H_0(-z)$ is a unique complementary filter of length N with respect to the filter $H_0(z)$.*

Proof. From Proposition 2.7, $\hat{X}(z) = \frac{X(z)[H_0(z)H_1(-z)-H_0(-z)H_1(z)]}{2}$. Since $H_1(z) = H_0(-z)$ in QMF, $\hat{X}(z) = \frac{X(z)[H_0(z)H_0(z)-H_0(-z)H_0(-z)]}{2}$. It is well known (see [1]) that given $a(x)$ and $b(x)$,

$$a(x)p(x) + b(x)q(x) = c(x)$$

has a solution $p(x)$ and $q(x)$ iff the greatest common divisor of $a(x)$ and $b(x)$ divides $c(x)$. Since QMFB has a PR property iff $\hat{X}(z) = kz^{-2l-1}X(z)$ for some constant k and some $l \in \mathbb{Z}$, the QMFB has a PR property iff the greatest common divisor of $H_0(z)$ and $H_0(-z)$ divides $2z^{-2l-1}$. Thus $H_0(-z)$ is a complementary filter of $H_0(z)$. Since $H_0(z)$ is symmetric and N is even, $H_0(-z)$ is antisymmetric. By (ii) in the proof of Theorem 3.3, $H_0(-z)$ is a unique complementary filter. ∎

Theorem 3.5. *Given a symmetric filter $H_0(z)$ of even length N and its complementary filter $H_1(z)$ of symmetric form and length N, all higher degree symmetric filters of length $N + 4m$ which are complementary to $H_0(z)$ are of the form*

$$G(z) = z^{-2m}H_1(z) + E(z)H_0(z),$$

where $E(z) = [a_1 + a_2z^{-2} + \cdots + a_mz^{-(2m-2)} + a_{m+1}z^{-2m} + a_mz^{-(2m+2)} + \cdots + a_2z^{-(4m-2)} + a_1z^{-4m}]$.

Proof. $E(z)H_0(z) = a_1h_0(0) + a_1h_0(1)z^{-1} + [a_1h_0(2) + a_2h_0(0)]z^{-2} + \cdots + a_1h_0(N-2)z^{-(N+4m-2)} + a_1h_0(N-1)z^{-(N+4m-1)}$. Thus $E(z)H_0(z)$ is symmetric about the point between $\frac{N+4m}{2}$ and $\frac{N+4m-2}{2}$. $H_1(z)z^{-2m} = h_1(0)z^{-2m} + h_1(1)z^{-(2m+1)} + h_1(2)z^{-(2m+2)} + \cdots + h_1(N-1)z^{-(N+2m-1)}$. Thus $H_1(z)z^{-2m}$ is symmetric about the point between $\frac{N+4m}{2}$ and $\frac{N+4m-2}{2}$. Hence $H_1(z)z^{-2m} + E(z)H_0(z)$ is symmetric. Since $H_1(z)$ is a complementary filter to $H_0(z)$, $z^{-2m}H_1(z)$ is a complementary filter to $H_0(z)$. It is easily checked from $E(z) = E(-z)$ that given $H_0(z)$, the solution of the equation

$$H_0(z)H_1(-z) - H_0(-z)H_1(z) = 0$$

is $H_1(z) = E(z)H_0(z)$. Thus $z^{-2m}H_1(z) + E(z)H_0(z)$ is a complementary filter of length $N + 4m$ to $H_0(z)$.

Suppose $G(z)$ is a complementary filter of length $N + 4m$ to $H_0(z)$. Then $H_0(z)G$
$(-z)$ is valid and of length $2N+4m-1$. Since $H_0(z)E(-z)H_0(-z)-H_0(-z)E(z)H_0(z)$

$= 0$, $H_0(z)E(z)H_0(-z)$ is valid and of length $2N+4m-1$. Let $K(z) = H_0(z)[G(-z) - E(z)H_0(-z)]$. Then $K(z)$ is valid and of length $2N+4m-1$. $K(z) = h_0(0)(g(0) - a_1h_0(0)) + [h_0(1)(g(0) - a_1h_0(0)) + h_0(0)(a_1h_0(1) - g(1))]z^{-1} + [h_0(2)(g(0) - a_1h_0(0)) + h_0(0)(g(2) - a_2h_0(0) - a_1h_0(2)) + h_0(1)(a_1h_0(1) - g(1))]z^{-2} + \cdots + [h_0(N-1)(a_1h_0(N-1) - g(N+4m-1))]z^{-(2N+4m-2)}$. By choosing $a_1 = \frac{g(0)}{h_0(0)}$, the coefficients of z^0 and $z^{-(2N+4m-2)}$ are zero. Since $H_0(z)G(-z)$ is valid and $a_1 = \frac{g(0)}{h_0(0)}$, the coefficients of z^{-1} and $z^{-(2N+4m-3)}$ are zero. Similarly, by choosing proper a_i for $i = 2, \ldots, m$, we can make the coefficients of four terms 0 for each a_i. By this process, $K(z)$ has length $2N-1$, has powers of z^{-1} in the range z^{-2m} to $z^{-2m+2N-2}$, and is valid. Since $K(z)$ has $H_0(z)$ as a factor and has z^{-2m} as a factor,

$$K(z) = z^{-2m}H_0(z)Q(z), \quad \text{where length of } Q(z) \text{ is } N.$$

By Proposition 3.3, the solution of length N is unique. Thus $K(z) = z^{-2m}H_0(z)H_1(-z)$. Hence $z^{-2m}H_0(z)H_1(-z) = H_0(z)(G(-z) - E(z)H_0(-z))$. That is, $G(-z) = z^{-2m}H_1$
$(-z) + E(z)H_0(-z)$. Thus $G(z) = z^{-2m}H_1(z) + E(z)H_0(z)$. ■

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