DERIVATIVES OF NEUMANN EIGENFUNCTIONS

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1. Introduction

Let $B(r)$ $(0 < r < \pi)$ be an n-dimensional spherical cap which is a geodesic ball, of radius r, in the unit n-sphere, and Δ the Laplace operator on the unit n-sphere. Let us consider the following Neumann eigenvalue problem:

$$\Delta f + \mu f = 0 \quad \text{in} \quad B(r), \qquad \frac{\partial f}{\partial n} = 0 \quad \text{on} \quad \partial B(r), \tag{1}$$

It is well-known that there is a discrete sequence of nonnegative real numbers

$$0 = \mu_0 < \mu_1 \leq \mu_2 \cdots \leq \mu_m \leq \cdots$$

satisfying (1).

In this note we are interested in the estimation of Neumann eigenvalues μ_i of a spherical cap $B(r)$.

As the counter part T. Matsuzawa, S. Tanno[6] and S. Sato[7] found the method of computing the first Dirichlet eigenvalue on a 2-dimensional spherical cap, which is effective under the condition $r \approx \pi$. The author[3] gave the estimation of the radial components of Dirichlet eigenfunctions on an n-dimensional spherical cap as follows:

Let λ be a Dirichlet eigenvalue of a spherical cap $B(r)$. Dirichlet eigenfunctions on a spherical cap $B(r)$ are of the form

$$f_\ell(n, t, \lambda) = (\sin t)^\ell \{1 + \sum_{j=1}^{\infty} K_j(\lambda) \sin^{2j}(t/2)\}, \tag{2}$$

where

$$K_j(\lambda) = 2^j \prod_{k=1}^{j} \frac{-\lambda + (k-1)(n+k-2)}{k(n+2k-2)}$$

(Refer to [1]). Since each Dirichlet eigenvalue λ of a spherical cap $B(r)$ satisfies $f_\ell(n,r,\lambda) = 0$, λ is a zero of the function $g(\lambda) = f_\ell(n,r,\lambda)$ given by the series of functions. If j goes to the infinity, $K_{j+1}(\lambda)/K_j(\lambda)$ does to 1. As j is larger, $\sum_{j=1}^{\infty} K_j(\lambda)\sin^{2j}(r/2)$ is almost equal to the geometric series with ratio $\sin^2(r/2)$. If we can obtain the error when we compute a zero of $g_N(\lambda) = 1 + \sum_{j=1}^{N} K_j(\lambda)\sin^{2j}(r/2)$, instead of a zero of $g(\lambda)$, explicit computation of the error is possible.

2. Neumann Eigenfunctions

The basic properties of Neumann eigenvalues on a spherical cap $B(r)$ are found in [4] pp. 39-52. Note that the derivative of the radial component of a Neumann eigenfunction on an odd dimensional spherical cap $B(r)$ is a rational function of finitely many trigometric functions (see [2]). Neumann eigenfunctions on a spherical cap $B(r)$ are the same form as (2). Simple computation shows that

$$\frac{\partial}{\partial t} f_\ell(n,r,\mu) = (\sin r)^{\ell-1}[\ell\cos r + \sum_{j=1}^{\infty} K_j(\mu)\sin^{2j}(r/2)((j+\ell)\cos r + j)]. \quad (3)$$

Since Neumann eigenvalue μ of a spherical cap $B(r)$ satisfies $\frac{\partial}{\partial t} f_\ell(n,r,\mu) = 0$, μ is a zero of the function $h(\mu)$ given by the series of functions, where

$$h(\mu) = \ell\cos r + \sum_{j=1}^{\infty} K_j(\mu)\sin^{2j}(r/2)((j+\ell)\cos r + j), \quad (4)$$

$$K_j(\mu) = 2^j \prod_{k=1}^{j} \frac{-\mu + \ell(n+\ell-1) + (k-1)(n+2\ell+k-2)}{k(n+2\ell+2k-2)}.$$

Consider the following ratio $R = R(j,\mu,r)$ similar to the case of Dirichlet eigenfunctions:

$$\sin^{2j+2}(r/2)K_{j+1}(\mu)((j+\ell+1)\cos r + j + 1)/\sin^{2j}(r/2)K_j(\mu)((j+\ell)\cos r + j) \quad (5)$$

$$= \sin^2(r/2)(K_{j+1}(\mu)/K_j(\mu))(1+\frac{\cos r+1}{(j+\ell)\cos r+j}).$$

If j goes to infinity, R does to $\sin^2(r/2)$. As j is larger,

$$\sum_{j=1}^{\infty} K_j(\mu)\sin^{2j}(r/2)((j+\ell)\cos r+j)$$

is almost equal to the geometric series with ratio $\sin^2(r/2)$. It seems that there is no simple relation between Neumann eigenvalues similar to [1] p.240, Proposition 1. It is clear that

$$(-n/2\mu)\frac{\partial}{\partial t}f_0(n,r,\mu) = \sin t f_0(n+2,\mu,r)$$

(see [2] p.4). If $\ell = 0$, the Neumann problem reduces to the Dirichlet problem, and then we can apply to the results of [3]. We may assume that $\ell \neq 0$. If we obtain the error when we compute a zero of

$$h_N(\mu) = \ell\cos r + \sum_{j=1}^{N} K_j(\mu)\sin^{2j}(r/2)((j+\ell)\cos r+j),$$

instead of a zero of $h(\mu)$, explicit computation of the error is possible. From now on, we consider an upper bound of $|h(\mu)-h_N(\mu)|$. Let

$$p(x) = \frac{-\mu+\ell(n+\ell-1)+x(n+2\ell+x-1)}{(x+1)(n+2\ell+2x)}.$$

If $n+2\ell > 4$ then $p(x)$ is decreasing, and if $n=2, \ell=1$ then $p(x)$ is increasing. Let $j_p(r)$ be the greatest integer less than or equal to $-\ell\cos r/(1+\cos r)$. Then $j_p(r)$ is nondecreasing as a function of r. If $j > j_p(r)$ then $(j+\ell)\cos r + j > 0$, and if $j \le j_p(r)$ then $(j+\ell)\cos r+j \le 0$. Let $j_q(\mu, r) = \min\{j' : |R| < 1 \textit{ for every } j \ge j'\}$. Note that $|R| \le \sin^2(r/2)|K_{j+1}(\mu)/K_j(\mu)|(1+1/(j-j_p(r)))$ if $j > j_p(r)$.

3. Conclusion

If $r \approx \pi$, the above computation is not effective, because $\sin^2(r/2) \approx 1$. It seems that there is no results about Dirichlet or Neumann eigenvalues of any dimensional spherical cap of radius $r \approx \pi$ except [5],[6],[7].

All results on the unit n-sphere can be also transferred into those of $S^n(\kappa)$ and $RP^n(\kappa)$ (n-sphere and real projective space of constant sectional curvature $\kappa > 0$), if r, λ, and μ are replaced by $r\sqrt{\kappa}$, λ/κ, and μ/κ, respectively.

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