방해물에 기인한 이층유체의 자유 계면에서의 변화 - Hydraulic Fall
Free surface flow of a Two-Layer fluid over a bump - Hydraulic Fall

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Abstract

We consider long nonlinear waves in the two-layer flow of an inviscid and incompressible fluid bounded above by a free surface and below by a rigid boundary. The flow is forced by a bump on the bottom. The derivation of the forced KdV equation fails when the density ratio h and the depth ratio ρ yields a condition $1+h\rho=(2-h)((1-h)^2+4\rho h)^{1/2}$. To overcome this difficulty we derive a forced modified KdV equation by a refined asymptotic method. Numerical solutions are given and hydraulic fall solution of a two layer fluid is expressed analytically in the case that derivation of the forced KdV(FKdV) equaition fails.

1 Introduction

We consider the waves between two immiscible, inviscid, and incompressible fluids of different but constant densities in the presence of small bump at the rigid bottom when the effect of gravity is considered (Fig. 1). We assume that the upper boundary is a free surface and the two dimensional bump is moving along the lower rigid boundary at a constant speed. By choosing a coordinate system moving with the object, the fluid motion becomes steady. Two critical speeds are obtained, near either one of which an FKdV for steady flow can be derived and has been studied extensively in [1] and [2]. Forbes [3],[4] studied steady flow of a two layer fluid over a bump numerically and found a hydraulic fall. Shen[5] used FKdV theory to find analytic expression of hydraulic fall of one layer fluid. An asymptotic approach for the case of a rigid upper boundary was developed without surface tension by Shen [6] on the basis of FKdV theory, and with surface tension by Choi, Sun, and Shen [7]. The case of free upper boundary was studied with surface tension by Choi, Sun, Shen [8] asymptotically on the basis of EKdV theory. In the case considered here, when the wave speed is near the smaller critical speed for internal wave, the nonlinear term in the FKdV may vanish and the derivation of FKdV fails. To overcome this difficulty, a refined asymtotic method is used to derive the Steady Modified KdV equation with forcing term (SFMKdV) in the following form:

$$(A\eta_2^2 + B)\eta_{2x} + C\eta_{2xxx} + Db_x = 0,$$

where A to D are constants depending on several parameters and $b(x)$ is a function with compact support due to the bump on the rigid lower boundary. We investigate solutions of the SFMKdV, which represent possible interfacial wave forms. By using this equation, we find the analytic expression of Forbes's hydraulic fall in the case that derivation of FKdV equation fails.

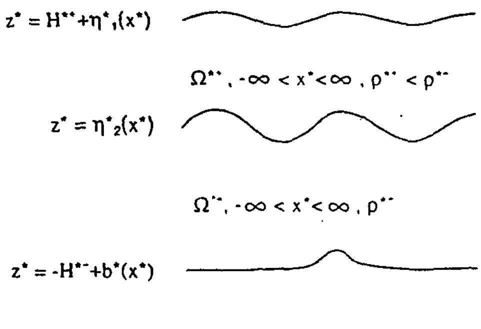

Fig. 1 Fluid Domain

2 Derivation of Steady Modified KdV equation with Forcing

We consider steady internal waves between two fluids. We assume that the two fluids are inviscid, incompressile and immiscible and have constant but different dinsities. The fluid domain is bounded above by a free surface and below by a horizontal rigid boundary with a bump. The domains of the upper fluid with constant density ρ^{*+} and the lower fluid with constant density ρ^{*-} are denoted by Ω^{*+} and Ω^{*-} respectively (Fig. 1). Assume that the small bump is moving with constant velocity C. In reference to a coordinate system moving with the bump, the flow is steady and moving with velocity C far upstream. Then the governing equations and boundary conditions are given by the Euler equations as follows:

In $\Omega^{*\pm}$,

$$\begin{aligned} u^{*\pm}_{x^*} + w^{*\pm}_{z^*} &= 0 \\ u^{*\pm}u^{*\pm}_{x^*} + w^{*\pm}u^{*\pm}_{z^*} &= -p^{*\pm}_{x^*}/\rho^{*\pm} \\ u^{*\pm}w^{*\pm}_{x^*} + w^{*\pm}w^{*\pm}_{z^*} &= -p^{*\pm}_{z^*}/\rho^{*\pm} - g\,; \end{aligned}$$

at the free surface, $z^* = H^{*+} + \eta_1^*$

$$u^{*+}\eta^*_{1x^*} - w^{*+} = 0$$
$$p^{*+} = 0\,;$$

at the interface, $z^* = \eta_2^*$

$$p^{*+} - p^{*-} = 0$$
$$u^{*\pm}\eta^*_{2x^*} - w^{*\pm} = 0\,;$$

at the rigid bottom, $z^* = -H^{*-} + b^*(x^*)$

$$w^{*-} - b^*_{x^*}u^{*-} = 0,$$

where $(u^{*\pm}, w^{*\pm})$ are horizontal and vertical velocities, $p^{*\pm}$ are pressures, $\rho^{*\pm}$ are densities of upper and lower fluids, and g is the gravitational acceleration constant. We define the following nondimensional variables:

$$\begin{aligned}
\epsilon &= H/L << 1,\\
\eta_1 &= \epsilon^{-1}\eta_1^*/H^{*-},\\
\eta_2 &= \epsilon^{-1}\eta_2^*/H^{*-},\\
p^{\pm} &= p^{*\pm}/gH^{*-}\rho^{*-},\\
(x,z) &= (\epsilon x^*, z^*)/H^{*-},\\
(u,w) &= (gH^{*-})^{-1/2}(u^*, \epsilon^{-1}w^*),\\
\rho &= \rho^{*+}/\rho^{*-} < 1,\\
U &= C/(gH^{*-})^{1/2},\\
h &= H^{*+}/H^{*-},\\
b(x) &= b^*(x)(H^{*-}\epsilon^3)^{-1},
\end{aligned}$$

where L is the horizontal scale, H is the vertical scale, $b(x) = b^*(x)(H^{*-}\epsilon^3)^{-1}$, H^{*+} and $-H^{*-}$ are the equilibrium depth of the upper and lower fluids at $x^* = -\infty$ respectively, and $z^* = -H^{*-} + b^*(x)$ is the equation for the bump. Then, in terms of them, the above equations become in $\Omega^{\pm}$,

$$\begin{aligned}
u^{\pm}_x + w^{\pm}_z &= 0 &(1)\\
u^{\pm}u^{\pm}_x + w^{\pm}u^{\pm}_z &= -p^{\pm}_x/\rho^{\pm} &(2)\\
\epsilon^2 u^{\pm}w^{\pm}_x + \epsilon^2 w^{\pm}w^{\pm}_z &= -p^{\pm}_z/\rho^{\pm} - 1 &(3)
\end{aligned}$$

at $z = h + \epsilon\eta_1$,

$$p^+ = 0, \qquad (4)$$
$$\epsilon u^+\eta_{1x} - w^+ = 0\,; \qquad (5)$$

at $z = \epsilon\eta_2$,

$$\epsilon u^-\eta_{2x} - w^- = 0 \qquad (6)$$
$$\epsilon u^+\eta_{2x} - w^+ = 0 \qquad (7)$$
$$p^+ - p^- = 0; \qquad (8)$$

at $z = -1$,

$$w^- = \epsilon^3 u b_x, \qquad (9)$$

where $\rho^+ = \rho < \rho^-, \rho^- = 1$ and $b(x)$ has a compact support.

Next, we use a unified asymptotic method to derive the equations of $\eta_1(x)$ and $\eta_2(x)$. We assume that velocity terms u, v and pressure term p have the following asymptotic expansions:

$$(u,w,p) = (u_0,w_0,p_0) + \epsilon(u_1,w_1,p_1) + \epsilon^2(u_2,w_2,p_2) + \cdots . \qquad (10)$$

By inserting (10) into nondimensionalized Euler equations and arranging the resulting equations according to the power of ϵ, one can easily see that $(u_0, 0, -\rho^{\pm} z + \rho h)$ are the solutions of the zeroth order system of equations and solve the first to third order system of differential equations[8]. It follows that $u_i^{\pm}, w_i^{\pm}, p_i^{\pm}, i = 1,2,3$, are all fuctions of $\eta_1(x)$ and $\eta_2(x)$ and the following equations of η_1 and η_2 are obtained.
At $z = h$,

$$\begin{aligned} & u_0 \eta_{1x} - w_1^+ \\ & + \epsilon(u_1^+ \eta_{1x} - \eta_1 w_{1z}^+ - w_2^+) \\ & + \epsilon^2(u_2^+ \eta_{1x} + \eta_1 \eta_{1x} u_{1z}^+ - w_{1zz}^+ \eta_1^2 \\ & - \eta_1 w_{2z}^+ - w_3^+) = 0, \end{aligned}$$

and at $z = 0$,

$$\begin{aligned} & u_0 \eta_{2x} - w_1^- \\ & + \epsilon(u_1^- \eta_{2x} - \eta_2 w_{1z}^- - w_2^-) \\ & + \epsilon^2(u_2^- \eta_{2x} + \eta_2 \eta_{2x} u_{1z}^- - w_{1zz}^- \eta_2^2 \\ & - \eta_2 w_{2z}^- - w_3^-) = 0. \end{aligned}$$

Then we use these equations to find the equations of the free surface $\eta_1(x)$ and the interface $\eta_2(x)$. By substituting $u_0, u_1^{\pm}, w_1^{\pm}, u_2^{\pm}, w_2^{\pm}, w_3^{\pm}$ into the above equations of η_1 and η_2, we can find the two relations between η_1 and η_2 and by solving η_1 for η_2, we obtain

$$\begin{aligned} & (u_0 - \rho c_1/u_0 - (1-\rho)/u_0)\eta_{2x} + \\ & \epsilon(E \eta_2 \eta_{2x}) + \epsilon^2(F_1 \eta_2^2 \eta_{2x} + F_2 \eta_{2x} \\ & + F_3 \eta_{2xxx} + F_4 b_x) = 0, \qquad (11) \end{aligned}$$

where, letting $\lambda = u_2^{\pm}(-\infty)$, $\eta_1(-\infty) = 0, \eta_2(-\infty) = 0$, $A = (2u_0^2 - (1-\rho))/(\rho + u_0^2 - h)$, $B = u_0/(\rho + u_0^2 - h)$, and $C = \rho A + 1 - \rho$,

$$\begin{aligned} E &= -C^2 u_0^{-3} - 2C u_0^{-1} \\ & \quad -\rho B(-2C - C^2 u_0^{-2} + 2A^2 \\ & \quad + hA^2 u_0^{-2} - 2A)/u_0^2, \\ F_1 &= -\rho(C^2/2 - 3C^3 u_0^{-2}/2 - C^2 \\ & \quad + A^3/2 + 3hA^3 u_0^{-2}/2 - A^2/2 \\ & \quad - A^2 + A^3)/u_0^4 \\ & \quad -3B\rho(u_0^{-1} + C u_0^{-3}) \\ & \quad (3(-A\rho - (1-\rho))/2 \\ & \quad -C^2 u_0^{-2}/2 + A^2/2 + hA^2 u_0^{-2}/2 \\ & \quad -A + A^2/2)/u_0 - 3C^2 u_0^{-3}/2 \\ & \quad -3C^3 u_0^{-5}/2, \\ F_2 &= (1 + C u_0^{-2})(\lambda - \rho B u_0^{-1}) \\ & \quad -\lambda \rho B u_0^{-1}(2 - A + \\ & \quad C u_0^{-2} - hA u_0^{-2}), \\ F_3 &= -\rho B u_0^{-1}(-A(\rho h^2/2 + \rho/3) \\ & \quad -(u_0^2 \rho h + (1-\rho)/3) \\ & \quad +A(\rho h^3/3)/\rho + u_0^2 h^2/2)/u_0 \\ & \quad -A(\rho h^2/2 + \rho/3) u_0^{-1} \\ & \quad -(u_0^2 \rho h + (1-\rho)/3) u_0^{-1}, \\ F_4 &= \rho B - u_0, \end{aligned}$$

From the zeroth order term of the equation (11), the critical speeds can be derived so that near them one has to develop a nonlinear theory for the motion of the interface and free surface of the given domain. $u_0-\rho A/u_0-(1-\rho)/u_0 = 0$ implies $u_0^4-(1+h)u_0^2+h(1-\rho)=0$, and $u_0^2 = 1+h\pm((1-h)^2+4\rho h)^{1/2}$. Next we consider the coefficients of $\eta\eta_x$ term in the first order terms of the equation (11). If E is not zero, the same result as in [1] and [5] can be carried out. In this article, Since we consider two layer fluid, we have the case that E vanishes if $u_0^2 = 1+h-((1-h)^2+4\rho h)^{1/2}$ and $1+h\rho = (2-h)((1-h)^2+4\rho h)^{1/2}$. With the conditions, we come up with the following time-independent modified KdV equation with forcing (SFMKdV),

$$F_1\eta_2^2\eta_{2x}+F_2\eta_{2x} + F_3\eta_{2xxx}+F_4b_x = 0. \quad (12)$$

where

$$\begin{aligned} F_1 &= 3u_0(4\rho+3h-u_0^2), \\ F_2 &= \lambda(2(1+h)u_0^2-4h(1-\rho))u_0^{-2}, \\ F_3 &= u_0^{-1}(h(1+h)/3 \\ &\quad -u_0^2(h^2+1+3\rho h)/3), \\ F_4 &= u_0(h-u_0^2). \end{aligned}$$

Since the sign of F_3F_1 determines the existence of solutions of (12) and only bounded solutions appeares without occurence of hydraulic fall solutions when $F_3F_1 > 0$ [7], we assume $F_3F_1 < 0$ in the following.

3 Hydraulic falls

We assume $U = u_0+\lambda\epsilon^2$ By dividing both sides of (12) by F_3, we have

$$\eta_{2xxx} = A_1\eta_2^2\eta_{2x}+A_2\eta_{2x}+A_3bx. \quad (13)$$

where

$$\begin{aligned} A_1 &= -F_1/F_3 > 0, \\ A_2 &= -F_2/F_3, \\ A_3 &= -F_4/F_3. \end{aligned}$$

If $\eta_2(x)$ is a solution of (13) and tends to 0 at $x=-\infty$ with $\eta_{2x}(-\infty)=0$, then, from $b(-\infty)=0$, η_2 satisfies

$$\begin{aligned} \eta_{2xx} &= A_1\eta_2^3/3+A_2\eta_2+A_3b(x), \quad (14) \\ \eta_2(-\infty) &= 0, \quad \eta_{2x}(-\infty)=0, \end{aligned}$$

We choose $\eta_2 \equiv 0$ in $(-\infty, x^-)$ and prove the existence of the solution of (14) on $[x^-, x^+]$, where $[x^-, x^+]$ is the compact support of $b(x)$. For that purpose, we define a complete metric space M and a closed ball B_r in M so that $M = \{\eta_2 \mid \eta_2 \in C([x^-, x^+]), \ \|\eta_2\| = \max_{x^-\le x\le x^+}|\eta_2(x)|\}$ $B_r = \{\eta_2 \in M \mid \|\eta_2\| \le r < \infty\}$. Then, by using contraction mapping theorem, the following theorem is proved [7].

Theorem 1

$$\eta_{2xx} - A_2\eta_2 - A_1\eta_2^3/3 = A_3 b(x),$$
$$x^- \le x \le x^+, \quad \|r\| \le M,$$
$$\eta_2(x^-) = \eta_{2x}(x^-) = 0$$

has a solution in $C^3(R)$ if $-A_2$ is sufficiently large.

Next we analyze the solutions of (14) ahead of bump. When $b(x) = 0$, we consider an initial value problem for (14) with initial value $\eta_2(x^+) = \alpha$, $\eta_{2x}(x^+) = \beta$. Then by integrating it from x^+ to $x > x^+$,

$$(\eta_{2x}(x))^2 = (A_1/6)\eta_2(x)^4 + A_2\eta(x)^2 + d$$

where $d = \beta^2 - (A_1/6)\alpha^4 - A_2\alpha^2$.(15)

If $\alpha = \beta = 0$, then (15) has the trivial solution $\eta(x) \equiv 0$. If $A_2^2 - 2A_1 d/3 > 0$,

$$(A_1/6) \quad \eta_2(x)^4 + A_2\eta_2(x)^2 + d = (A_1/6)(\eta_2^2 - \xi_0)(\eta_2^2 - \xi_1)$$

where

$$\xi_0 = -3A_2/A_1 + 3(A_2^2 - 2A_1 d/3)^{1/2}/A_1,$$
$$\xi_1 = -3A_2/A_1 - 3(A_2^2 - 2A_1 d/3)^{1/2}/A_1.$$

Hence, the solutions of (15) in this case are the following periodic wave solutions.

When $\xi_0 > \xi_1 > 0$, $\eta_2(x) = \xi_1^{1/2}\sin\phi$, where

$$(A_1\xi_0/6)^{1/2}(x - x^+) = \int_{\phi_0}^{\phi}(1 - k^2\sin\theta)^{-1/2}d\theta,$$
$$\phi_0 = \sin^{-1}(\alpha\xi_1^{-1/2}),$$
$$\text{and} \quad k^2 = \xi_1/\xi_0 < 1.$$

When $\xi_0 > 0 > \xi_1$, $\eta_2(x) = \xi_0^{1/2}\cos\phi$, where

$$\gamma(x - x^+) = \int_{\phi_0}^{\phi}(1 - k^2\sin\theta)^{-1/2}d\theta,$$
$$\gamma = (A_1(\xi_0 - \xi_1)/6)^{1/2},$$
$$\phi_0 = \cos^{-1}(\alpha\xi_0^{-1/2}),$$
$$\text{and} \quad k^2 = \xi_0/(\xi_0 - \xi_1) < 1.$$

If α and β are not zeros and $A_2^2 - 2A_1 d/3 = 0$, then (15) has a solution $\eta_2(x) = \pm(-3A_2/A_1)^{1/2}\tanh(-A_2 x/2)$. If $A_2^2 < 2A_1 d/3$, $\eta_{2x}(x) = \pm(d_1 + (A_1/6)(\eta_2^2(x) + c_1)^2)^{1/2}$ for some $d_1 > 0$ and the solution is unbounded.

Having shown the existence of a solution of (14) from x^- to x^+ and analyzing the solutions ahead and behind the bump, we now find the global solution of (14), numerically using Runge-Kutta Method. The numerical results are given in Fig. 2 to 6. Fig. 2 shows typical hydraulic fall solution profile of (14) and we give solution curve of hydraulic falls in Fig. 3. In both numerical results, we assume $x^- = -1, x^+ = 1$, and $b(x) = R\sqrt{1-x^2}$ for $|x| \le 1$ and $b(x) = 0$ elsewhere. As λ being increased from $-\infty$, symmetric wave-free solutions are embedded in periodic wave solutions for discrete values of λ for which $\eta_2(x^+) = \eta_{2x}(x^+) = 0$. In Fig. 4 and 5, we present typical symmetric wave-free solution and Fig. 6 gives the solution profile of symmetric wave solutions with one hump. As λ being increased and converging to a value, above which solution diverges, the period of the periodic solution becomes larger and hydraulic fall solution appears at the limiting value of λ as $A_2^2 - 2A_1 d/3$ becomes zero. Hence we have found the analytic expression of hydraulic fall solution $\eta_2(x) = (-3A_2/A_1)^{1/2}\tanh(-A_2(x - x_0)/2)$, where x_0 is a phase shift. We note that in the case of $1 + h\rho \neq (2-h)((1-h)^2 + 4\rho h)^{1/2}$ FKdV theory can be used to find the expression of hydraulic fall and same result as in [5] can be carried out.

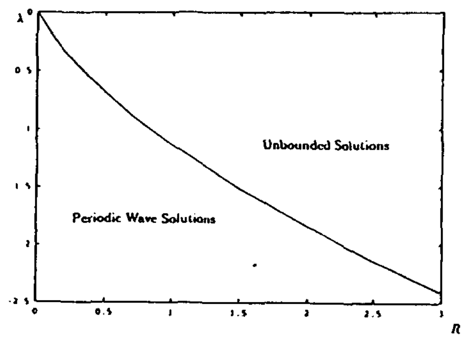

Fig. 3. Solution curve of hydraulic fall, $h = 0.5$

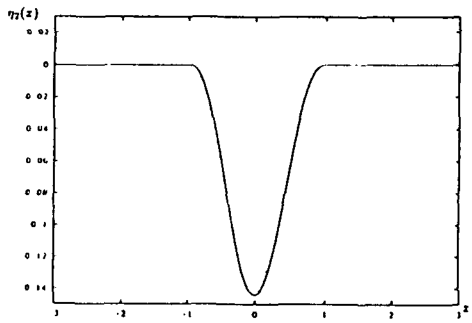

Fig. 4. Symmetric wave solution with one hump
$h = 0.5, R = 1, \lambda = -2.73397$

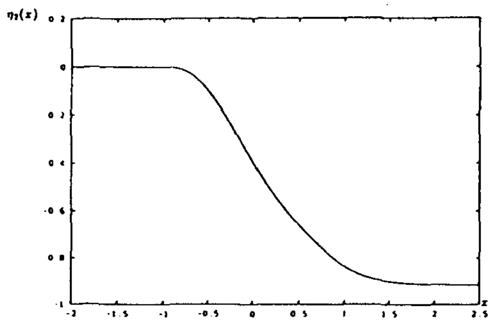

Fig. 2. Hydraulic fall solution, $h = 0.5, R = 1, \lambda = -1.13911$

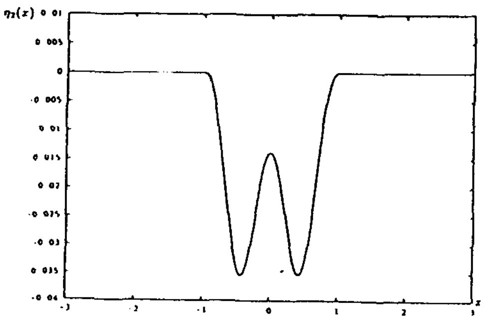

Fig. 5. Symmetric wave solution with two humps
$h = 0.5, R = 1, \lambda = -9.06601$

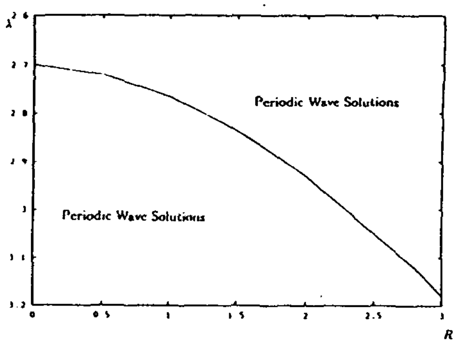

Fig. 6. Solution curve of Symmetric wave solutions with one hump, $h = 0.5$

4 Conclusion

We consider the physical problem of steady state flow past a positive, symmetric body at the horizontal bottom of a two-layer fluid. The derivation of Forced K-dV equation fails when $1 + h\rho = (2 - h)((1 - h)^2 + 4\rho h)^{1/2}$, and the Forced Modified KdV equation is derived by a unified asymptotic analysis. Two parameters, λ and R, appear in the equation and can affect its solution behavior. λ is a deviation of the flow speed at the flow speed at far upstream from the critical speed u_0, and R is the hight of the bump. We study mathematically different types of period-free solutions which may appear in different regions of the λ and R. This investigation may help us understand the flow pattern under parameter change in a two-layer fluid with bump at the rigid boundary. In particular, we have found the analytic expression of the hydraulic fall which was found numerically by Forbes [3].

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