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ON ASYMPTOTIC STABILITY FOR PERTURBED DIFFERENTIAL EQUATION

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ABSTRACT. In this paper, we investigated several asymptotic stability properties of the system for the type of $\frac{dy}{dt} = h(t)^{-1}F(t, k(t)y(t))$.

1. Introduction

Recently, many authors - T. Taniguchi[10], S. K. Chang, H. J. Lee, and Y. S. Oh[2, 3], etc. have tried to generalize Perron's celebrated theorem.

Nowadays, various stability problems are happening in mechanics, electronics, control engineering, economics, physics, chemistry, biology, and lots of practical problems in education, circumstance, and society. And too many authors have been studying to solve them and presenting numerous properties. Among such qualitative theorems, Perron's stability theorem is very popular and important.

In a sense, it is the fact that, for stable linear differential equations, the perturbed linear differential equations by very small perturbations are also stable.

However, when linear differential equations are considered, are their perturbed differential equations stable if some perturbations would be given ?

It is an important and interesting problem under some conditions for the perturbations which the qualitative properties of the original equations are preserved or improved in a suitable sense.

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In [9], T. Taniguchi obtained various stability theorems of perturbed differential equations and gave a partial answer for this question and generalized Perron's stability Theorem by the variation of parameters formula and comparison theorems of integral inequalities.

After that, S.K. Chang, H.J. Lee, and Y.S. Oh [2,3] also defined notions of stability, which are called $T(\mu)$-stability and $\varphi(t)$- stability, etc., of the trivial solution for systems of differential equations. And they investigated generalizing Perron's stability theorem .

2. Preliminaries and Definitions

Throughout this paper, R^n be the n-dimensional Euclidean space and $R^+ = [0, \infty)$.

For a given fuction $g(t, y) \in C[R^+ \times R^n, R^n]$, we consider following differential equation:

$$\frac{dy}{dt} = g(t, y) \tag{1}$$

Let us assume that $g(t, 0) = 0$ for all $t \in R^+$, and the equation (1) is well-posed for sufficiently small initial values at any initial time.

$C[X, Y]$ denotes the set of all continuous mappings from a topological space X to a topological space Y. Set $B_\delta = \{x \in R^n : \|x\| < \delta\}$ for a positive real number δ where $\|\cdot\|$ denotes a usual norm. Let $y(t) \equiv y(t; t_1, y_1)$ denote a solution of (1) with an initial value (t_1, y_1), and we assume the existence and the uniqueness of the solutions for the equation (1) with sufficiently small initial values.

Now we introduce generalized definitions of stability for the equation (1).

Definition 2.1. ([2]) Let $\varphi(t)$ be a continuous positive real function on R^+. The trivial solution of (1) is said to be *$\varphi(t)$-stable* if for any $\varepsilon > 0$ and for any $t_1 \geq 0$, there exists $\delta(t_1, \varepsilon) > 0$ such that if $\|y(t_1)\| < \delta(t_1, \varepsilon)$, then $\|y(t)\varphi(t)^{-1}\| < \varepsilon$ for all $t \geq t_1$; *$\varphi(t)$-quasi-asymptotically stable* if for any $\varepsilon > 0$ and for any $t_1 \geq 0$, there exist $\delta(t_1) > 0$ and $T(t_1, \varepsilon) > 0$ such that if $\|y(t_1)\| < \delta(t_1)$, then $\|y(t)\varphi(t)^{-1}\| < \varepsilon$ for all $t \geq t_1 + T(t_1, \varepsilon)$; *$\varphi(t)$-asymptotically stable* if it is $\varphi(t)$-stable and $\varphi(t)$-quasi-asymptotically stable.

Other stability concepts can be similarly defined.(See [2, 3, 10])

Let $A(t)$ be a continuous $n \times n$ matrix defined on R^+ and let $f(t,x) \in C[R^+ \times R^n, R^n]$ with $f(t,0) = 0$ for any $t \in R^+$.

Consider a linear differential equation

$$\frac{dx}{dt} = A(t)x \tag{2}$$

and a perturbed differential equation of (2)

$$\frac{dx}{dt} = A(t)x + f(t,x). \tag{3}$$

Let $U(t)$ be the fundamental matrix solution of (2). Then the solution $x(t)$ of (3) satisfies the well-known integral equation

$$x(t) = U(t)U^{-1}(t_1)x(t_1) + \int_{t_1}^{t} U(t)U^{-1}(s)f(s,x(s))ds, \ t \geq t_1. \tag{4}$$

Next, the following lemmas play important roles for our theorems in the next section.

Lemma 2.2. ([2]) *Let $k(t)$ be a continuous positive real function on R^+. Then the trivial solution of the differential equation* (2) *is $k(t)$-stable if and only if $\|U(t)U^{-1}(s)\| \leq k(t)h(s)^{-1}$, $t \geq s \geq 0$ for a continuous positive real function $h(t)$ on R^+.*

Lemma 2.3. ([9]) *Let us $k \in C[R^+,(0,\infty)]$. Then*

(1) *if $k(t)$ is bounded, then $k(t)$-stability of the trivial solution of (1) implies the stability of the trivial soultion of (1);*

(2) *if $k(t) \geq 1$, then the stability of the trivial solution of (1) implies $k(t)$-stability of the trivial soution of (1);*

(3) *if $k(t)$ is constant on R^+, then the stability and $k(t)$-stability of (1) are equivalent.*

3. Main results

In this section, we discuss asymptotic stability of the trivial solution for the perturbed differential equation (3).

Assume that $f(t,0) \equiv 0$ throughout this section.

Let us consider the following differential equation

$$\frac{dy}{dt} = h(t)^{-1}F(t, k(t)y(t)) \tag{5}$$

where $k(t)$ and $h(t)$ are continuous positive real functions on R^+.

Henceforth, we assume that the above differential equations (5) possess the existence and the uniqueness of solutions on R^+ with sufficiently small initial values.

We are now in a position to prove our results.

Theorem 3.1. *Let the following conditions hold for the differential equation* (3)*:*

(1a) $\|f(t,x)\| \leq F(t, \|x\|)$, $F(t,0) \equiv 0$, *and* $F(t,u)$ *is monotone nondecreasing with respect to* u *for each fixed* $t \geq 0$,

(1b) $F(t,u) \in C[R^+ \times R^+, R^+]$,

(1c) *the trivial solution of the differential equation* (2) *is* $k(t)$*-stable for a continuous positive real function* $k(t)$ *on* R^+.

If the trivial solution of (5) *is* $\varphi(t)$*-asymptotically stable for a continuous positive real function* $\varphi(t)$ *on* R^+, *then the trivial solution of* (3) *is* $k(t)\varphi(t)$*-asymptotically stable.*

Proof. First, we prove that the trivial solution of (3) is $k(t)\varphi(t)$-stable for a continuous positive real function $\varphi(t)$ on R^+.

Let $x(t) \equiv x(t; t_1, x_1)$ be a solution of (3) with an initial value (t_1, x_1), $t_1 \geq 0$. Then the solution $x(t)$ is of the following form (4).
Accordingly, we obtain that from conditions (1a) and (1c),

$$\|x(t)\| \leq k(t)h(t_1)^{-1}\|x_1\| + k(t)\int_{t_1}^{t} h(s)^{-1}F(s, \|x(s)\|)ds.$$

Thus, let $y(t) \equiv y(t; t_1, y_1)$ be the solution of (5) passing through (t_1, x_1) and let $h(t_1)^{-1}\|x_1\| < y_1$. Then
$k(t)^{-1}\|x(t)\| - \int_{t_1}^{t} h(s)^{-1}F(s, \|x(s)\|)ds \leq h(t_1)^{-1}\|x_1\| < y_1$.
While, since $y_1 = y(t) - \int_{t_1}^{t} h(s)^{-1}F(s, k(s)y(s))ds$,

$$\begin{aligned} &\|x(t)\| - k(t)\int_{t_1}^{t} h(s)^{-1}F(s, \|x(s)\|)ds \\ &\qquad < k(t)y(t) - k(t)\int_{t_1}^{t} h(s)^{-1}F(s, k(s)y(s))ds. \end{aligned}$$

Therefore, applying [2, Lemma 2.2], we obtain that $\|x(t)\| < k(t)y(t)$ and $\|y(t)\varphi(t)^{-1}\| < \varepsilon$ for all $t \geq t_1$, which completes the proof.

Next, suppose that the trivial solution of (5) is $\varphi(t)$-quasi-asymptotically stable for a continuous positive real function $\varphi(t)$ on R^+. Then for any $\varepsilon > 0$ and any $t_1 \geq 0$, there exist $\delta_1(t_1) > 0$ and $T(t_1, \varepsilon) > 0$ such that if $\|y(t_1)\| < \delta_1(t_1)$, then $\|y(t)\varphi(t)^{-1}\| < \varepsilon$ for all $t \geq t_1 + T(t_1, \varepsilon)$. Thus set $\delta(t_1) = h(t_1)\delta_1(t_1)$. If $\|x(t_1)\| < \delta(t_1)$, then we can take $y_1 > 0$ such that $h(t_1)^{-1}\|x(t_1)\| < y_1 < \delta_1(t_1)$. Accordingly, $\|x(t)\| < k(t)y(t)$ for all $t \geq t_1 + T(t_1, \varepsilon)$ and $\|y(t)\varphi(t)^{-1}\| < \varepsilon$ for all $t \geq t_1 + T(t_1, \varepsilon)$. Therefore, we have $\|x(t)(k(t)\varphi(t))^{-1}\| < \varepsilon$ for all $t \geq t_1 + T(t_1, \varepsilon)$, which completes the proof of Theorem 3.1. □

Theorem 3.2. *Let the conditions* (1a), (1*b*), *and* (1c) *hold for the differential equation* (3) *and let the trivial solution of the differential equation* (5) *be stable.*

If $k(t) \to 0$ *as* $t \to \infty$ *(in* (1c) *and* (5)*), then the trivial solution of* (3) *is asymptotically stable.*

Proof. Let $x(t) \equiv x(t; t_1, x_1)$ and $y(t) \equiv y(t; t_1, y_1)$ be solutions of (3) and (5), respectively. By the hypothesis, since the trivial solution of (5) is stable, for any fixed $B > 0$ and any $t_1 \geq 0$, there exists $\delta(t_1, B) > 0$ such that if $\|y(t_1)\| < \delta(t_1, B)$, then $\|y(t)\| < B$ for all $t \geq t_1$. Set $\delta(t_1) \equiv h(t_1)\delta(t_1, B)$. If $\|x(t_1)\| < \delta(t_1)$, then we take $y_1 > 0$ such that $h(t_1)^{-1}\|x(t_1)\| < y_1 < \delta(t_1, B)$. From the proof of theorem 3.1, we have $\|x(t)\| < k(t)y(t)$ for all $t \geq t_1$ and $\|y(t)\| < B$ for all $t \geq t_1$. Thus, $\delta(t_1, B) + B > \|y(t_1)\| + \|y(t)\| \geq \int_{t_1}^t h(s)^{-1}F(s, k(s)y(s))ds$. Since

$$\begin{aligned}\|x(t)\| - k(t)\int_{t_1}^t h(s)^{-1}F(s, \|x(s)\|)ds \\ < k(t)y(t) - k(t)\int_{t_1}^t h(s)^{-1}F(s, k(s)y(s))ds,\end{aligned}$$

we have

$$\begin{aligned}\int_{t_1}^t h(s)^{-1}F(s, k(s)y(s))ds &= \Big|\int_{t_1}^t h(s)^{-1}F(s, k(s)y(s))ds\Big| \\ &\leq \|y(t)\| + \Big|k(t)^{-1}\|x(t)\| - \int_{t_1}^t h(s)^{-1}F(s, \|x(s)\|)ds\Big| \\ &< \|y(t)\| + \|y(t_1)\| \\ &< B + \delta(t_1, B).\end{aligned}$$

While, since $\|x(t)\| < k(t)y(t)$ for all $t \geq t_1$,

$$\int_{t_1}^{t} h(s)^{-1}F(s, \|x(s)\|)ds \leq \int_{t_1}^{t} h(s)^{-1}F(s, k(s)y(s))ds,$$

which implies that for any $\varepsilon > 0$ and any $t_1 \geq 0$, there exists $T \equiv T(\varepsilon, t_1, x(t; t_1, x_1)) > 0$ such that $\int_T^t h(s)^{-1}F(s, \|x(s)\|)ds < \varepsilon$ for all $t \geq T$. Thus, we obtain that from (4) and the condition (1a),

$$\begin{aligned}\|x(t)\| \leq \|U(t)\|\|U^{-1}(t_1)x_1\| + \|U(t)\|\|\int_{t_1}^{T} U^{-1}(s)f(s, x(s))ds\| \\ + k(t)\int_{T}^{t} h(s)^{-1}F(s, \|x(s)\|)ds.\end{aligned}$$

Since $k(t) \to 0$ as $t \to \infty$, from the condition (1c), $U(t) \to 0$ as $t \to \infty$, which implies that $\|x(t)\| \to 0$ as $t \to \infty$.

Hence the trivial solution of (3) is quasi-asymptotically stable. And, from [2, Theorem 3.1] , since the trivial solution of (3) is $k(t)$-stable for a positive real function $k(t)$, the trivial solution of (3) is stable, which completes the proof of the theorem. □

From the above theorem, we obtain the following corollary.

Corollary 3.3. *Let the conditions* (1a), (1b), *and* (1c) *hold for the differential equation* (3) *and let the trivial solution of the differential equation* (5) *be stable.*

If $k(t) = Ke^{-\mu t}$ *(in* (1c) *and* (5)*) for positive real number* μ, *then the trivial solution of* (3) *is asymptotically stable.*

Remark. See[2] for the elegant and suitable example of Theorem 3.2.

REFERENCES

1. F. Brauer and J.A. Nohel, *Qulitative theory of ordinary differential equations*, Benjamin, New York, 1969.
2. S. K. Chang, H. J. Lee, and Y. S. Oh, *A grneralization of Perron's stability theorems for perturbed linear differential equations*, Kyungpook Math. J. **33** (1993), 2, 163-171.
3. S. K. Chang, Y. S. Oh and H. J. Lee, *Exponential behavior of solutions for perturbed linear ordinary differential equations*, Kyungpook Math. J. **32** (1992), 3, 311-320.
4. S. K. Choi, K. S. Koo and K. H. Lee, *Lipschitz stability and exponential asymptotic stability in perturbed systems*, J. Korean Math. Soc. **29** (1992), 1, 175-190.
5. A. Halanay, *Differential equations: stability*, Oscillation, Time Lag. Academic Press, New York, 1966.

6. V. Lakshmikantham and S. Leela, *Differential and integral inequalities Theory and Appl. Vol. I, Academic Press, New York, 1969.*
7. Y. S. Oh and J. H. An, *On uniformly stability for nonlinear differential systems*, to appear.
8. Y. S. Oh, J. H. An and S. M. Hwang, *On stability of the perturbed linear differential equations*, Youngnam Math. J. **1** (1994).
9. Y. S. Oh, J. S. Yi, S. M. Hwang and J. H. An, *On some asymptotic stability theorems of perturbed linear differential equations*, to appear.
10. D. T. Taniguchi, *Stability theorems of perturbed linear ordinary differential equations*, J. Math. Anal. and Appl. **149** (1990), 583-598.

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