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ON $\phi(t)$-STABILITY FOR THE COMPARISON DIFFERENTIAL SYSTEM

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ABSTRACT. We obtained sufficient conditions for $\phi(t)$-stability and uniform $\phi(t)$-stability of the trivial solution of comparison differential system. we also investigated the corresponding stability concepts of the trivial solution of the differential system using the thoery of differential inequlities through cones and the method of cone-valued Lyapunov functions.

1. Introduction

Let R^n denote the n-dimensional Euclidean space with any convenient norm $\|\cdot\|$, and scalar product (,). $R_+ = [0, \infty)$, $R = (-\infty, \infty)$, $R^n_+ = \{U \in R^n : U_i \geq 0, i = 1, 2, \cdots, n\}$, $C[R_+ \times R^n, R^n]$ denotes the space of continuous functions mapping $R_+ \times R^n$ into R^n.

Definition 1.1. A proper subset K of R^n is called a *cone* if (i) $\lambda K \subset K$, $\lambda \geq 0$; (ii) $K + K \subset K$; (iii) $K = \overline{K}$; (iv) $K^\circ \neq \emptyset$; (v) $K \cap (-K) = \{0\}$, where $\overline{K}$ and K° denote the closure and interior of K, respectively, and ∂K denotes the boundary of K. The order relation on R^n induced by the cone K is defined as follows :

Let $x, y \in K$, then $x \leq_K y$ iff $y - x \in K$ and $x <_{K^\circ} y$ iff $y - x \in K^\circ$.

Definition 1.2. The set K^* is called the *adjoint cone* if $K^* = \{\phi \in R^n : (\phi, x) \geq 0,$ for all $x \in K\}$ satisfies properties (i) $\sim$ (v) of Definition 1.1.

Consider the differential system

$$x' = f(t, x), \quad x(t_0) = x_0, t_0 \geq 0 \tag{1}$$

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where $f \in C[R_+ \times R^N, R^N]$. Define S_ρ by $S_\rho = \{x \in R^N : \|x\| < \rho, \rho > 0\}$. Let $K \subset R^n$ be a cone in R^n , $n \leq N$, and $V \in C[R_+ \times S_\rho, K]$. Define for $(t,x) \in R_+ \times S_\rho, h > 0$, the function $D^+V(t,x)$ by $D^+V(t,x) = \limsup_{h \to 0^+}(\frac{1}{h})[V(t+h, x+hf(t,x)) - V(t,x)]$.

Consider the comparison differential system

$$u' = g(t,u), \quad u(t_0) = u_0, t_0 \geq 0 \tag{2}$$

where $g \in C[R_+ \times K, R^n]$, and K is a cone in R^n.

Let $S(\rho) = \{u \in K : \|u\| < \rho, \rho > 0\}$, $v \in C[R_+ \times S(\rho), K]$ and define for $(t,u) \in R_+ \times S(\rho)$, $h > 0$, the function $D^+v(t,u)$ by $D^+v(t,u) = \limsup_{h \to 0^+}(\frac{1}{h})[v(t+h, u+hg(t,u)) - v(t,u)]$.

Now we give $\phi(t)$-stability definition of the trivial solution of (2).

Definition 1.3. The trivial solution $u = 0$ of (2) is

$\phi(t)$-equistable if, for each $\varepsilon > 0$, $t_0 \in R_+$, there exists a positive function $\delta = \delta(t_0, \varepsilon)$ that is continuous in t_o for each ε such that the inequality $(\phi(t_0), u_0) < \delta$ implies $(\phi(t), r(t)) < \varepsilon, t \geq t_0$.

Other $\phi(t)$-stability concepts can be similarly defined.

In Definition 1.3, and for the rest of this paper, $r(t)$ denotes the maximal solution of (2) relative to the cone $K \subset R^n$. We denote

$\mathcal{P} = \{a \in C[R_+, R_+] : a(u)$ is strictly increasing in u and $a(0) = 0\}$.

The following comparison theorem plays a prominent role whenever we employ cone-valued Lyapunov functions.

Lemma 1.4. ([6]) *Assume that*

(i) $v \in C[R_+ \times S(\rho), K]$ *and* $v(t,u)$ *satisfies a local Lipschitz condition in* x *relative to* K *and for* $(t,u) \in R_+ \times S(\rho)$, $D^+v(t,u) \leq_K g(t, v(t,u))$;

(ii) $g \in C[R_+ \times K, R^n]$ *and* $g(t,u)$ *is quasimonotone in* u *with respect to* K *for each* $t \in R_+$.

If $r(t,t_0,u_0)$ *is the maximal solution of* (2) : $u' = g(t,u)$, $u(t_0) = u_0$ *relative to* K *and* $x(t;t_0,x_0)$ *is any solution of* (1) : $x' = f(t,x)$ *such that* $V(t_0,x_0) \leq_K u_0$, *then on the common interval of existence, we have* $V(t, x(t,t_0,x_0)) \leq_K r(t,t_0,u_0)$.

In general, ϕ_0-stability of the trivial solution $u = 0$ of (2) implies $\phi(t)$-equistability of the trivial solution $u = 0$ of (2).

In particular, if $\phi(t)$ in ${K_0}^*$ is constant , then $\phi(t)$-equistability and ϕ_0-equistability of the trivial solution $u = 0$ of (2) are equivalent.

Other stability notion can be similarly defined.(See [5])

2. Main results

Theorem 2.1. *Assume that*

(*i*) $v \in C[R_+ \times S(\rho), K]$, $v(t,0) = 0$, $v(t,u)$ *is locally Lipschitzian in* u *relative to* K, *and for each* $(t,u) \in R_+ \times S(\rho)$, $D^+v(t,u) \leq_K 0$.

(*ii*) $g \in C[R_+ \times K, R^n]$, $g(t,0) = 0$, $g(t,u)$ *is quasimonotone in* u *relative to* K.

(*iii*) *For some bounded continuous function* $\phi(t) \in {K_0}^*$ *and* $(t,u) \in R_+ \times S(\rho)$, $a[(\phi(t), r(t))] \leq (\phi(t), v(t,u))$, *for some differentiable function* $a \in \mathcal{P}$, $t \geq t_0$.

Then the trivial solution $u = 0$ *of* (2) *is* $\phi(t)$*-equistable.*

Proof. Since $v(t,0) = 0$ and $v(t,u_0)$ is continuous in t_0, given $a(\varepsilon) > 0$, $t_0 \in R_+$, there exists $\delta_1 = \delta_1(t, a(\varepsilon))$ such that $\|u_0\| < \delta_1$ implies $\|v(t_0,u_0)\| < a(\varepsilon)$, $a \in \mathcal{P}$.

Now for some bounded continuous function $\phi(t) \in {K_0}^*$, for each $t \geq t_0$, $\|\phi(t_0)\| \cdot \|u_0\| < \|\phi(t_0)\|\delta_1$ implies $\|\phi(t)\|\|v(t_0,u_0)\| < \|\phi(t)\|a(\varepsilon)$. Thus $(\phi(t_0), u_0) < \|\phi(t_0)\|$ δ_1 implies $(\phi(t), v(t_0,u_0)) < \|\phi(t)\| \cdot a(\varepsilon)$. Put $\delta = \|\phi(t_0)\|\delta_1$ and put $M = \sup\{\|\phi(t)\| | t \geq t_0\}$. Then $(\phi(t_0), u_0) < \delta$ implies $(\phi(t), v(t_0,u_0)) \leq \|\phi(t)\| \cdot \|v(t_0,u_0)\|$ $< M \cdot a(\varepsilon)$. Let $u(t)$ be any solution of (2) such that $(\phi(t_0), u_0) < \delta_0$. Then by (i), $v(t,u) \leq_K v(t_0,u_0)$, $t \geq t_0$. Thus $(\phi(t_0), u(t_0)) < \delta$ implies $a[(\phi(t), r(t))] < (\phi(t), v(t_0,u_0)) < M \cdot a(\varepsilon)$.

Hence it is not difficult to prove that $(\phi(t), r(t)) < k\varepsilon$ for all $t \geq t_0$ by using the mean value theorem, which completes the proof. □

Theorem 2.2. *Assume that for some* $\phi(t) \in C[R^+, K^*]$

(*i*) $v \in C[R_+ \times S(\rho), K]$, $v(t,0) = 0$, $v(t,u)$ *is locally Lipschitzian in* u *relative to* K, *for each* $(t,u) \in R_+ \times S(\rho)$.

(*ii*) $g \in C[R_+ \times K, R^n]$, $g(t,0) = 0$, $g(t,u)$ *is quasimonotone in* u *relative to* K.

(*iii*) *For some continuous function* $\phi(t) \in {K_0}^*$ *with* $\|\phi(t)\| \longrightarrow \infty$ *as* $t \to \infty$, $t \geq t_0$ *and* $(t,u) \in R_+ \times S(\rho)$, $D^+(\phi(t), v(t,u)) \leq 0$ *and* $(\phi(t), r(t)) \leq b(\phi(t), v(t,u))$ *for some* $b \in \mathcal{P}$, $t \geq t_0$.

Then the trivial solution $u = 0$ of (2) is $\phi(t)$-equistable.

Proof. Since $v(t,0) = 0$ and $v(t_0, u_0)$ is continuous in t_0, given $b^{-1}(\varepsilon) > 0$, $t_0 \in R_+$, there exists δ_1 such that $\|u_0\| < \delta_1$ implies $v(t_0, u_0) < b^{-1}(\varepsilon)$. For each $t \geq t_0$, $\|\phi(t_0)\| \cdot \|u(t_0)\| < \|\phi(t_0)\|\delta_1$ implies $(\phi(t_0), v(t_0, u_0)) < \|\phi(t_0)\| \cdot b^{-1}(\varepsilon)$. Thus $(\phi(t_0), u_0) < \|\phi(t_0)\|\delta_1$ implies $(\phi(t_0), v(t_0, u_0)) < \|\phi(t_0)\| \cdot b^{-1}(\varepsilon)$.

Let $u(t)$ be any solution of (2) such that $(\phi(t_0), u_0) < \delta$. Put $\delta = \|\phi(t_0)\|\delta_1$. Thus if $(\phi(t_0), u_0) < \delta$, then $(\phi(t), r(t)) \leq b(\phi(t), v(t,u)) \leq b(\phi(t_0), v(t_0, u_0)) < b(\|\phi(t_0)\| \cdot b^{-1}(\varepsilon))$. By using the mean value theorem,we have $(\phi(t), r(t)) \leq k\varepsilon$ for each $t \geq t_0$,the proof is complete. □

Theorem 2.3. *Assume that*

(*i*) $v \in C[R_+ \times S(\rho), K]$, $v(t,0) = 0$, *$v(t,u)$ is locally Lipschitzan in u relative to K, and for each $(t,u) \in R_+ \times S(\rho)$, and for some bounded $\phi(t) \in {K_0}^*$* $D^+(\phi(t), v(t,u)) \leq 0$.

(*ii*) $g \in C[R_+ \times K, R^n]$, $g(t,0) = 0$, *$g(t,u)$ is quasimonotone in u relative to K.*

(*iii*) *For this $\phi(t) \in {K_0}^*$ and $(t,u) \in R_+ \times S(\rho)$,*

$$a[(\phi(t), r(t))] \leq (\phi(t), v(t,u)) \leq b[(\phi(t), r(t))],\ a, b \in \mathcal{P}.$$

Then the trivial solution $u = 0$ of (2) is uniformly $\phi(t)$-stable.

Proof. For $\varepsilon > 0$, let $\delta = b^{-1}[a(\varepsilon)]$ independent of t_0 for $a, b \in \mathcal{P}$. Let $u(t)$ be any solution of (2) such that $(\phi(t_0), u_0) < \delta$. Then by hypothesis, $(\phi(t), v(t,u))$ is decreasing and so

$$(\phi(t), v(t,u)) \leq (\phi(t_0), v(t_0, u_0)).$$

Thus $a[(\phi(t), r(t))] \leq (\phi(t_0), v(t_0, u_0)) \leq b[(\phi(t), r(t))] < b(\delta) = a(\varepsilon)$. Hence

$$(\phi(t_0), u(t_0)) < \delta \text{ implies } (\phi(t), r(t)) < \varepsilon. \quad \square$$

Theorem 2.4. *Assume that*

(*i*) $v \in C[R_+ \times S(\rho), K]$, *$v(t,u)$ is locally Lipschitzian in u relative to K and for* $(t,u) \in R_+ \times S_\rho$, $D^+ v(t,u) \leq_K g(t, v(t,u))$,

(*ii*) $g \in C[R_+ \times K, R^n]$ *and $g(t,u)$ is quasimonotone in u relative to K for each* $t \in R_+$,

(*iii*) $f(t,0) = 0$, $g(t,0) = 0$, $(t,x) \in R_+ \times S_\rho$, $b(\|x\|) \leq (\phi(t), V(t,x)) \leq a(t, \|x\|), a, b \in \mathcal{P}$, *for some* $\phi(t) \in {K_0}^*$.

If the trivial solution $u = 0$ of (2) is $\phi(t)$-equistable, then the trivial solution $x = 0$ of (1) is equistable.

Proof. Let $0 < \varepsilon < \rho$ and $t_0 \in R_+$. Suppose that the trivial solution $u = 0$ of (2) is $\phi(t)$-equistable. Then given $b(\varepsilon) > 0$, $t_0 \in R_+$, there exists $\delta = \delta(t_0, \varepsilon) > 0$ such that $(\phi(t_0), u_0) < \delta$ implies $(\phi(t), r(t)) < b(\varepsilon), t \geq t_0$. Choose $a(t_0, \|x_0\|) = (\phi(t_0), u(t_0))$. Then $(\phi(t_0), V(t_0, x_0)) \leq a(t_0, \|x_0\|) = (\phi(t_0), u(t_0))$ implies $V(t_0, x_0) \leq_K u_0$ because $u_0 = u(t_0)$.

Let $x(t, t_0, x_0)$ be any solution of (1) such that $V(t_0, x_0) \leq_K u_0$. Then by Lemma 1.4, $V(t, x) \leq_K r(t)$.

Now choose $\delta_1 > 0$ such that $a(t_0, \delta_1) = \delta$. Thus the inequalities $\|x_0\| < \delta_1$ and $a(t_0, \|x_0\|) < \delta$ hold simultaneously. Thus, since $b(\|x\|) \leq (\phi(t), V(t, x)) \leq (\phi(t), r(t)) < b(\varepsilon)$. $\|x(t; t_0, x_0)\| < \varepsilon$ whenever $\|x_0\| < \delta_1$. Hence the trivial solution $x = 0$ of (1) is equistable.

Theorem 2.5. *Let the conditons (i) ,(ii), and(iii) of theorem 2.4 hold. If the trivial solution $u = 0$ of (2) is uniformly $\phi(t)$-stable, then the trivial solution $x = 0$ of (1) is uniformly stable.*

Theorem 2.6. *Let the conditons (i) and (ii) of theorem 2.4 hold. Assume further that for $c > 0$, $d > 0$, $(\phi(t_0), u_0) \leq \|x_0\|^d$ and $c\|x\|^d \leq (\phi(t), V(t, x))$. If the trivial solution $u = 0$ of (2) is exponentially asymptotically $\phi(t)$-stable, then the trivial solution $x = 0$ of (1) is exponentially asymptotically stable.*

Proof. Let $x(t, t_0, x_0)$ be any solution of (1) such that $V(t_0, x_0) \leq_K u_0$. Then by Lemma 1.4, we have that $V(t, x) \leq_K r(t)$. Thus $c\|x\|^d \leq (\phi(t), V(t, x)) \leq (\phi(t), r(t))$. Since the trivial solution $u = 0$ of (2) is exponentially asymptotically $\phi(t)$-stable, then there exists $\sigma > 0$, $\alpha > 0$ both real numbers such that $(\phi(t), r(t)) \leq \sigma(\phi(t_0), u(t_0)) \exp[-\alpha(t - t_0)]$, $t \geq t_0$ and $c\|x\|^d \leq \sigma(\phi(t_0), u(t_0)) \exp[-\alpha(t - t_0)] \leq \sigma\|x_0\|^d \cdot \exp[-\alpha(t - t_0)]$. This implies that $\|x\| \leq M \cdot \|x_0\| \exp[-\beta(t - t_0)]$, $t \geq t_0$ where $M = \sigma/c$, $\alpha/d = \beta$. □

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