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ON ESTIMATION OF ROOT BOUNDS OF POLYNOMIALS

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ABSTRACT. In this work we will show that,in the sense of the Maximum overestimation factor, the absolute root bound functional derived from the new formula for the divided difference is better than the other known results in this area.

1. Introduction

Many researches have been done to estimate the magnitude of the changes of the roots from the perturbed polynomials. For more informations and references on such works, see [6,9,10,12,13]. Before proceeding,we will give short comments on notations and some known results from the theory of divided differences. The most detailed exposition of their properties can be found in Mile-Thomson[7].

Definition 1.1. Let $p(z)$ be a polynomial in the complex variable z. The first divided difference of $p(z)$ is denoted by the relation

$$p[z_0, z_1] = \frac{p(z_0) - p(z_1)}{z_0 - z_1}.$$

The n-th divided difference is defined by the induction in terms of the (n-1)-th one by the formula

$$p[z_0, \ldots, z_n] = \frac{p[z_0, \ldots, z_{n-2}, z_n] - p[z_0, \ldots, z_{n-2}, z_{n-1}]}{z_n - z_{n-1}}. \tag{1.1}$$

In order to derive a new formula for the divided difference which is useful in studying perturbation of roots, we need the following lemma.

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Lemma 1.2 [3,7].

$$p[z_0,\dots,z_n]=\frac{1}{2\pi i}\int_\Gamma \frac{p(z)}{(z-z_0)\cdots(z-z_n)}\,dz,$$

where the points $z_0,\dots,z_n$ lie inside the contour Γ.

If $p(z)$ is a polynomial of degree n, then by Newton's interpolation formula, $p(z)$ can be reconstructed uniquely from the values of the divided differences at $z_0,\dots,z_n$ as follows:

$$p(z)=p[z_0]+p[z_0,z_1](z-z_0)+\cdots+p[z_0,\dots,z_n](z-z_0)\cdots(z-z_{n-1}).$$

For more informations and references to these discoveries, see [3,7].

Let $p(z)$ be a polynomial of degree n. We denote the set of roots of $p(z)$ by $Q_n=\{q_1,\dots,q_n\}$, the letters $\alpha,\beta,\gamma,\dots$ will represent subsets of Q_n, and $|\alpha|,|\beta|,|\gamma|,\dots$ the number of elements in these subsets. For $\alpha\subseteq Q_n$ we denote by $p[\alpha]$ the divided difference of $p(z)$, calculated at the points $q_i\in\alpha$. If $\alpha=\emptyset$, then $p[\alpha]=0$. If $\alpha,\beta,\gamma,\dots$ are subsets of Q_n, then we shall denote by $\alpha',\beta',\gamma',\dots$ complements of these subsets in Q_n. For any $\alpha\subseteq Q_n$, we set

$$(z-q)^\alpha=\prod_{q_i\in\alpha}(z-q_i),\qquad (z-q)^\alpha=1\ for\ \alpha=\emptyset,$$

$$(q-\tilde{q})^\alpha=\prod_{q_i\in\alpha}(q_i-\tilde{q}_i),\qquad (q-\tilde{q})^\alpha=1\ for\ \alpha=\emptyset.$$

Remark 1.3. If a polynomial $p(z)$ has multiple roots, then each root must be counted in the set Q_n as many times as its multiplicity. In this case, any subset α of Q_n may contain some copies of this multiple roots, while all other copies of these multiple roots will be contained in the complement α'.

Remark 1.4. Let $Q_n=\{q_1,\dots,q_n\}$ be a fixed set. Then for any subset $\{q_i,q_j,q_k,\dots\}\subseteq Q_n$, we shall always set $i<j<k<\cdots$ throughout this paper.

Now we will define $(q_\alpha-\tilde{q})^\nu$ as follows; for any subset $\alpha\subseteq\beta$ such that $|\beta|=m\le n$, set $\alpha=\{q_{\alpha_1},q_{\alpha_2},\cdots\}$, $\beta'=\{q_{c_1},q_{c_2},\cdots\}\subseteq Q_n$. Choose $\nu=\{q_{c_{j_1}},q_{c_{j_2}},\dots,q_{c_{j_{|\nu|}}}\}\subseteq\beta'$ so that $|\nu|=n+1-m-|\alpha|$, then we define

$$(q_\alpha-\tilde{q})^\nu=(q_{\alpha_{i_1}}-\tilde{q}_{c_{j_1}})(q_{\alpha_{i_2}}-\tilde{q}_{c_{j_2}})\cdots(q_{\alpha_{i_{|\nu|}}}-\tilde{q}_{c_{j_{|\nu|}}})$$

so that $i_1 = j_1,\ i_2 = j_2 - 1, \dots, i_{|\nu|} = j_{|\nu|} - |\nu| + 1$. We also set for any $\alpha, \nu \subseteq Q_n$,

$$(q_\alpha - \tilde{q})^\nu = \begin{cases} 1 & \text{for } |\nu| = 0 \\ 0 & \text{for } |\nu| < 0. \end{cases}$$

The next result is the new formula for the divided difference for $r[\beta]$ which is basic to the results in this paper. See [8,9] for more details.

Theorem 1.5. *Suppose that* $p(z) = (z - q_1)\cdots(z - q_n)$, *deg* $r(z) \leq n - 1$, $p(z) + r(z) = (z - \tilde{q}_1)\cdots(z - \tilde{q}_n)$ *and* $Q_n = \{q_1, \dots, q_n\}$. *Then for any subset* $\beta \subseteq Q_n$ *with* $|\beta| = m \leq n$, *we have*

$$r[\beta] = \frac{1}{2\pi i}\int_\Gamma \frac{(z - \tilde{q}_1)\cdots(z - \tilde{q}_n)}{(z - q)^\beta}\,dz \tag{1.2}$$

$$= \sum_{\substack{|\alpha| \geq 1 \\ \alpha \subseteq \beta}} (q - \tilde{q})^\alpha \sum_{\substack{\nu \subseteq \beta' \\ |\nu| = n+1-m-|\alpha|}} (q_\alpha - \tilde{q})^\nu.$$

Now we will define universal polynomials $P_\beta(\rho, |q_i - q_j|)$ from Theorem 1.5.

Definition 1.6. Let $Q_n = \{q_1, \dots, q_n\}$, $\beta \subseteq Q_n$ and $\rho \geq 0$. From the above formula, we will define universal polynomials $P_\beta(\rho, |q_i - q_j|)$ satisfying some specific properties given by Tulovsky[10,11]:

$$P_\beta(\rho, |q_i - q_j|) = \sum_{\substack{|\alpha| \geq 1 \\ \alpha \subseteq \beta}} \rho^{|\alpha|} \sum_{\substack{\nu \subseteq \beta' \\ |\nu| = n+1-m-|\alpha|}} (q_\alpha - q)^\nu_\rho,$$

where $(q_\alpha - q)^\nu_\rho = (|q_{\alpha_{i_1}} - q_{c_{j_1}}| + \rho)\cdots(|q_{\alpha_{i_{|\nu|}}} - q_{c_{j_{|\nu|}}}| + \rho)$.

Theorem 1.7. *Let* $p(z) = (z - q_1)\cdots(z - q_n)$, $r(z)$ *be a polynomial with degree* $\leq n - 1$ *and* $\tilde{p}(z) = p(z) + r(z) = (z - \tilde{q}_1)\cdots(z - \tilde{q}_n)$.

For a given $\rho \geq 0$, *if* $|q_i - \tilde{q}_i| \leq \rho$ *for all* i, *then for non-empty subset* $\beta \subseteq Q_n$,

$$|r[\beta]| \leq P_\beta(\rho, |q_i - q_j|) \ . \tag{1.3}$$

For more informations and references to these results, see[9,10].

2. Preliminaries

Let us begin by introducing some definitions and well known results on this field. These are taken mainly from A.Van der Sluis[12]. For $p(z) = z^n + b_{n-1}z^{n-1} + \cdots + b_1 z + b_0$ with roots $q_1, \ldots, q_n$, $U(p)$ is defined by

$$U(p) = max\{|q_i|\}.$$

Ξ will denote the class of monic complex polynomials of degree n. A root-bound functional (rbf) on Ξ will be a real functional M such that $M(p) \geq U(p)$ for all $p(z) \in \Xi$. A root bound (rb) for $p(z)$ will be a real number m such that $m \geq U(p)$.

Definition 2.1. A rbf M on Ξ such that $M(p) = M(\tilde{p})$ whenever $p(z) = z^n + b_{n-1}z^{n-1} + \cdots + b_1 z + b_0$, $\tilde{p}(z) = z^n + c_{n-1}z^{n-1} + \cdots + c_1 z + c_0$ with $|c_i| = |b_i|, 0 \leq i \leq n-1$, is called an absolute rbf on Ξ.

Now, some results on this field will be presented without proof. For the proofs and references see Van der Sluis[12].

Lemma 2.2. *(Cauchy's Theorem [6]). Let $\tilde{p}(z) = z^n + b_{n-1}z^{n-1} + \cdots + b_1 z + b_0$. If $k(n)$ is the positive solution of the equation*

$$K^n - |b_{n-1}|K^{n-1} - \cdots - |b_1|K - |b_0| = 0, \tag{2.1}$$

then all roots of $\tilde{p}(z)$ lie in the disk $B(0, k(n))$.

Remark 2.3. *[6,12]. By Cauchy's Theorem, the unique positive root z_0 of $z^n - |b_{n-1}|z^{n-1} - \cdots - |b_0| = 0$ is an absolute root bound functional. For any $p(z)(\neq z^n) \in \Xi$, we denote the corresponding z_0 as $B(p) = z_0$ and also define $B(z^n) = 0$, then B is the best absolute rbf of all absolute rbfs. While B is optimal, the positive root z_0 of the equation (2.1) can't be easily calculated.*

Therefore, other more computable absolute rbfs are widely used. Next we give examples of absolute rbfs which are well-known from the literature.

Let $p(z) = z^n + b_{n-1}z^{n-1} + \cdots + b_1 z + b_0$.

$$Q(p) = \begin{cases} \sum_{i=0}^{n-1} |b_i| & \text{if } \sum_{i=0}^{n-1} |b_i| \geq 1 \\ \sqrt[n]{\sum_{i=0}^{n-1} |b_i|} & \text{if } \sum_{i=0}^{n-1} |b_i| \leq 1. \end{cases} \tag{1}$$

$$R(p) = min\,(R_1(p), R_2(p))\,, \tag{2}$$

$$R_1(p) = max\left(1, \sum_{i=0}^{n-1} |b_i|\right),\ R_2(p) = max\,(1+|b_1|, \dots, 1+|b_{n-1}|, |b_0|,)\,.$$

$$S(p) = 2max\left\{|b_{n-1}|, \sqrt{|b_{n-2}|}, \dots, \sqrt[n-1]{|b_1|}, \sqrt[n]{\frac{|b_0|}{2}}\right\}. \tag{3}$$

$$T(p) = max\left\{2|b_{n-1}|, 2\frac{|b_{n-2}|}{|b_{n-1}|}, \dots, 2\frac{|b_1|}{|b_2|}, \frac{|b_0|}{|b_1|},\right\}. \tag{4}$$

Remark 2.4. *Van der Sluis [12] showed that for the absolute rbf S, $S(p) \le 2B(p)$ for all $p(z) \in \Xi$ and hence S is nearly optimal among all absolute rbfs.*

3. Root bound of Polynomials

We will now show how a special case of Theorem 1.7 leads to an absolute rbf that in some cases gives better estimates than the classical absolute rbfs listed in the previous example.

Theorem 3.1. *Let $p(z) = (z - z_0)^n, r(z)$ be a polynomial of degree $\le n-1$ and $\tilde{p}(z) = p(z) + r(z) = (z - \tilde{q}_1)\cdots(z - \tilde{q}_n)$. If $|z_0 - \tilde{q}_i| \le \rho$ for all i, then for $\beta = \{z_0, \dots, z_0\}$ with $|\beta| = m \le n$, we have*

$$|r[\beta]| \le \binom{n}{n+1-m}\rho^{n+1-m}. \tag{3.1}$$

Conversely if (3.1) holds for all $|\beta| \neq 0$, then we have

i). $|z_0 - \tilde{q}_i| \le \tilde{K}(n)\rho$, where $\tilde{K}(n) = \dfrac{1}{\sqrt[n]{2}-1}$,

ii). $\dfrac{1}{\sqrt[n]{2}-1} = \dfrac{n}{ln2} - \epsilon_n$, $0 < \epsilon_n < 0.5$ for $n \ge 2$.

Proof. **i).** Suppose $|z_0 - \tilde{q}_i| \le \rho$ for all i. For any $\beta = \{z_0, \dots, z_0\}$ with $|\beta| = m \le n$, from the properties of our universal polynomial $P_\beta(\rho, |q_i - q_j|)$, obviously we have

$$|r[\beta]| \le \binom{n}{n+1-m}\rho^{n+1-m}.$$

Let us assume (3.1) holds. let $G = \{z : |z - z_0| \leq k\rho\}$ with boundary Γ. By Newton's interpolation formula, we have

$$r(z) = r[z_0] + r[z_0, z_0](z - z_0) + \cdots + r[z_0, \ldots, z_0](z - z_0)^{n-1}.$$

For any $z \in \Gamma$, we get the following inequality

$$|r(z)| \leq \rho^n + \binom{n}{n-1}\rho^{n-1}|z - z_0| + \cdots + \binom{n}{1}\rho|z - z_0|^{n-1}.$$

So $\dfrac{|r(z)|}{|p(z)|} \leq \dfrac{\sum_{i=0}^{n-1}\binom{n}{n-i}k^i}{k^n}$. Now, we need to find the positive solution $\tilde{K}(n)$ of the equation;

$$K^n - \sum_{i=0}^{n-1}\binom{n}{n-i}k^i = 0.$$

Then clearly we can see that each $\tilde{q}_i$ lies in the circle $|z - z_0| \leq \tilde{K}(n)\rho$. The binomial formula gives that

$$K^n - \sum_{i=0}^{n-1}\binom{n}{n-i}k^i = k^n - \{(k+1)^n - k^n\} = 2k^n - (k+1)^n = 0. \tag{3.2}$$

Then the positive solution of (3.2) is $\tilde{K}(n) = \dfrac{1}{\sqrt[n]{2} - 1}$, and we have $|z_0 - \tilde{q}_i| \leq \tilde{K}(n)\rho$.

ii). From $\tilde{K}(n) = \dfrac{1}{\sqrt[n]{2} - 1}$, we have $\sqrt[n]{2} = 1 + \dfrac{1}{\tilde{K}(n)}$. Taylor's Theorem gives

$$\frac{1}{\tilde{K}(n)} - \frac{1}{2\tilde{K}^2(n)} < \frac{ln2}{n} = ln(1 + \frac{1}{\tilde{k}(n)}) < \frac{1}{\tilde{K}(n)} \quad \text{for} \quad n \geq 2.$$

Then we can easily check that $\dfrac{n}{ln2} - \dfrac{1}{2} < \tilde{K}(n) < \dfrac{n}{ln2}$. That is, $\tilde{K}(n) = \dfrac{n}{ln2} - \epsilon_n, \quad 0 < \epsilon_n < 0.5 \quad$ for $n \geq 2$.

Now consider $p(z) = z^n + b_{n-1}z^{n-1} + \cdots + b_1 z + b_0$ with roots $q_1, \ldots, q_n$. From the Theorem 3.1, if $|b_{n-i}| \leq \binom{n}{i}\rho^i$ for all i, then we get the estimate $|q_i| \leq \dfrac{\rho}{\sqrt[n]{2} - 1}$ for all i. From the fact that $p(z)$ and $z^m p(z)$ have the same roots except 0 for any non-negative integer m, we may now consider $z^m p(z) = z^m(z^n + b_{n-1}z^{n-1} + \cdots + b_1 z + b_0)$. Since $\binom{n}{j} \leq \binom{n+m}{j}, m \geq 0$,

$$|b_{n-j}| \leq \binom{n+m}{j}\rho^j = \frac{(n+m)(n+m-1)\cdots(n+m-j+1)}{j!}\rho^j.$$

So,

$$\left(\frac{j!|b_{n-j}|}{(n+m)(n+m-1)\cdots(n+m-j+1)}\right)^{\frac{1}{j}} \leq \rho.$$

Set $t = n+m$. Then

$$\tilde{K}(t)\rho \geq \frac{\sqrt[j]{j!|b_{n-j}|}}{(\sqrt[t]{2}-1)\{t(t-1)\cdots(t-j+1)\}^{\frac{1}{j}}}.$$

By using L'hôpital's rule, we have

$$\lim_{t\to\infty}(\sqrt[t]{2}-1)\{t(t-1)\cdots(t-j+1)\}^{\frac{1}{j}} = ln2.$$

So we have the following inequality

$$\lim_{t\to\infty}\tilde{K}(t)\rho \geq \frac{\sqrt[j]{j!|b_{n-j}|}}{ln2}, \qquad j = 1,\ldots,n. \tag{3.3}$$

That is, we conclude that all roots of $p(z) = z^n + b_{n-1}z^{n-1} + \cdots + b_1z + b_0$ lie in

$$|z| \leq \frac{1}{ln2}\max_{1\leq j\leq n}\sqrt[j]{j!|b_{n-j}|}.$$

So the functional

$$H(p) = \frac{1}{ln2}\max_{1\leq j\leq n}\sqrt[j]{j!|b_{n-j}|}$$

is an absolute rbf.

First of all, we give some definitions and some known results on the maximum overestimation factor from Van der Sluis[12].

Definition 3.2. For $p(z) \in \Xi$ and $r > 0$, we will denote the polynomial defined by $p^r(z) = r^n p(z/r)$. A rbf $M : \Xi \to R$ is called homogeneous if $M(p^r(z)) = rM(p(z))$ for $p(z) \in \Xi$ and $r > 0$. M is called normal if i).M is a continuous rbf and ii).$M(p^r(z))$ is an increasing function of $r > 0$ for which $M(p^r(z)) > inf(M(\Xi))$.

Definition 3.3. $Mof_M(t) = \dfrac{t}{inf\{U(p) : p(z) \in \Xi, M(p) = t\}}$ is called the maximum overestimation factor. (Note that if M is a homogeneous rbf on Ξ, then $Mof_M(t)$ is independent of t. In this case we will write Mof.)

Note. If $p(z) \in \Xi$ and $M(p) = t$, then $Mof_M(t) \geq \dfrac{t}{U(p)}$. So one obtains the lower bound $U(p) \geq \dfrac{t}{Mof_M(t)}$ whenever $M(p) = t$.

Lemma 3.4. *(Van der Sluis[12])*

1). *For any normal absolute rbf $M, Mof_M(t) \geq Mof_B(t)$ for the best absolute rbf $B(p)$.*

2). *If M is a homogeneous normal rbf on Ξ, then $Mof_M = M(p(z))$ for $p(z) = z^n + \binom{n}{1} z^{n-1} + \cdots + \binom{n}{n-1} z + \binom{n}{n} = (z+1)^n$.*

3). *i) $Mof_B = \dfrac{1}{\sqrt[n]{2} - 1} \approx \dfrac{n}{ln2} \approx 1.4n$, $B(p)$ is homogeneous normal. ii) $Mof_S = 2n$, $S(p)$ is homogeneous normal.*

Applying Lemma 3.4 to $H(p)$ we obtain the following our results.

Theorem 3.5. *i).$H(p)$ is homogeneous normal. ii). $Mof_H = \dfrac{n}{ln2} \approx Mof_B$ for $n \geq 1$.*

As measured by the maximum overestimation factor, $H(p)$ performs better than $S(p)$. Moreover $H(p)$ and $B(p)$ perform similarly in this measure. Note that while B(p) is the best absolute rbf, B(p) can't be easily calculated. See Remark 2.3 for more detail. Now we give an example where $H(p)$ gives much better estimate than $S(p)$.

Consider $p(z) = z^4 - 16z^3 - 146z^2 - 271z - 109$. From the equation $K^4 - 16K^3 - 146K^2 - 271K - 109 = 0$, we have

$$H(p) \approx 24.65, \qquad B(p) = 22.901, \qquad S(p) = 32,$$

$$Q(p) = 542, \qquad R(p) = 272, \qquad T(p) = 32.$$

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