J. Korea Soc. Math. Ed. Series B: The Pure and Applied Mathematics
4, No. 1 (1997) pp. 47–60

THE PARTIAL DIFFERENTIAL EQUATION ON FUNCTION SPACE WITH RESPECT TO AN INTEGRAL EQUATION

SEUNG JUN CHANG AND SANG DEOK LEE

ABSTRACT. In the theory of the conditional Wiener integral, the integrand is a functional of the standard Wiener process. In this paper we consider a conditional function space integral for functionals of more general stochastic process and the generalized Kac-Feynman integral equation. We first show that the existence of a partial differential equation. We then show that the generalized Kac-Feynman integral equation is equivalent to the partial differential equation.

1. Introduction

Let $(C_0[0,T], \mathcal{B}(C_0[0,T]), m_w)$ denote Wiener space where $C_0[0,T]$ is the space of all continuous functions x on $[0,T]$ with $x(0) = 0$. Many physical problem can be formulated in terms of the conditional Wiener integral $E[F|X]$ of the functional defined on $C_0[0,T]$ of the form

$$F(x) = \exp\{-\int_0^t \theta(s, x(s))ds\} \tag{1.1}$$

where $X(x) = x(t)$ and $\theta(\cdot,\cdot)$ is a sufficiently smooth function on $[0,T] \times \mathbb{R}$. It is indeed known from a theorem of Kac [11] that the function $U(\cdot,\cdot)$ defined on $[0,T] \times \mathbb{R}$ by

$$U(t,\xi) = \frac{1}{\sqrt{2\pi t}} \exp\{-\frac{(\xi-\xi_0)^2}{2t}\} E[F(x(t)+\xi_0)|x(t) = \xi - \xi_0] \tag{1.2}$$

is the solution of the partial differential equation

$$\frac{\partial U}{\partial t} = \frac{1}{2}\frac{\partial^2 U}{\partial \xi^2} - \theta U \tag{1.3}$$

Received by the editors April 30, 1997.
1991 *Mathematics Subject Classification.* 60J65.
Key words and phrases. Generalized Brownian motion process, Function space integral, Coditional function space integral, Kac-Feynman integral equation.

satisfying the condition $U(0,\xi)=\delta(\xi-\xi_0)$. In [9], Donsker and Lions showed that the function

$$U(t,\xi)=E[\delta_{t,\xi-\xi_0}(x)F(x)] \tag{1.4}$$

is the solution of the partial differential equation (1.3) where $\delta_{t,\xi}$ $(t>0,\xi\in\mathbb{R})$ is the Donsker's delta function formally defined by

$$\delta_{t,\xi}(x)=\frac{1}{2\pi}\int_{\mathbb{R}} e^{iu(x(t)-\xi)}du, \quad x\in C_0[0,T].$$

In [18], in order to provide a rigorous treatment of the function (1.4) involving the Donsker's delta function, Yeh introduced the concept of the conditional Wiener integral and derived a Fourier inversion formula for conditional Wiener integrals :

$$\begin{aligned} E[F|x(t)=\xi]\frac{1}{\sqrt{2\pi t}}\exp\{-\frac{\xi^2}{2t}\} \\ =\frac{1}{2\pi}\int_{\mathbb{R}} e^{-iu\xi}E[e^{iux(t)}F]du, \quad \xi\in\mathbb{R} \end{aligned} \tag{1.5}$$

which gives a formula to obtain the explicit evaluation of the solution of the partial differential equation (1.3).

Using the inversion formula (1.5), Yeh [18] derived the Kac-Feynman equation for a time independent continuous potential function $\theta(\xi)$. In [7], Chung and Kang, using the Yeh's inversion formula, obtained similar results for a time dependent bounded potential $\theta(s,\xi)$. In [15], Skoug and Park obtained a simple formula for expressing conditional Wiener integrals with a vector-valued conditioning function in terms of ordinary Wiener integral, and then used the formula to derive the Kac-Feynman integral equation for a time independent potential function $\theta(\xi)$.

In this paper we extend the ideas of [7,11,14] from the Wiener processes to more general stochastic processes. We note that the Wiener process is free of drift and is stationary in time. However, the stochastic process considered in this paper is a process subject to drift and is nonstationary in time.

In Section 3, under the appropriate regularity conditions on $\theta(\cdot,\cdot)$, $a(\cdot)$ and $b(\cdot)$, the function U given by (2.3) has the first and second partial derivatives with respect to η. In Section 4, it has been shown that the generalized Kac-Feynman integral equation (2.3) is equivalent to a partial differential equation which is generalized form of the equation (1.3).

2. Preliminaries

Let $D = [0, T]$ and let $(\Omega, \mathcal{B}, P)$ be a probability measure space. A real valued stochastic process X on $(\Omega, \mathcal{B}, P)$ and D is called *a generalized Brownian motion process* if $X(0, \omega)=0$ almost everywhere and for $0 \le t_0 < t_1 < \cdots < t_n \le T$, the n-dimensional random vector $(X(t_1, \omega), \cdots, X(t_n, \omega))$ is normally distributed with the density function

$$K(t, \vec{\eta}) = \Big((2\pi)^n \prod_{j=1}^{n} (b(t_j) - b(t_{j-1}))\Big)^{-1/2} \cdot \exp\Big\{ -\frac{1}{2} \sum_{j=1}^{n} \frac{((\eta_j - a(t_j)) - (\eta_{j-1} - a(t_{j-1})))^2}{b(t_j) - b(t_{j-1})} \Big\}, \tag{2.1}$$

where $\vec{\eta} = (\eta_1, \cdots, \eta_n)$, $\eta_0 = 0$ and $a(t)$ is a real valued function with $a(0) = 0$ and $b(t)$ is a strictly increasing real valued function with $b(0) = 0$.

As explained in [16, p.18-20], X induces a probability measure μ on the measurable space $(\mathbb{R}^D, \mathcal{B}^D)$ where $\mathbb{R}^D$ is the space of all real valued functions $x(t)$, $t \in D$, and $\mathcal{B}^D$ is the smallest σ-algebra of subsets of $\mathbb{R}^D$ with respect to which all the coordinate evaluation maps $e_t(x) = x(t)$ defined on $\mathbb{R}^D$ are measurable. The triple $(\mathbb{R}^D, \mathcal{B}^D, \mu)$ is a probability measure space. This measure space is called *the function space induced by the generalized Brownian motion process X determined by $a(\cdot)$ and $b(\cdot)$*.

Let X be an $\mathbb{R}^n$-valued measurable function and Y a complex valued μ-integrable function on $(\mathbb{R}^D, \mathcal{B}^D, \mu)$. Let $\mathcal{F}(X)$ denote the σ-algebra of subsets of $\mathbb{R}^D$ generated by X. Then by the definition of conditional expectation, the conditional expctation of Y given $\mathcal{F}(X)$, written $E[Y|X]$, is any $\mathbb{R}^n$-valued $\mathcal{F}(X)$-measurable function on $\mathbb{R}^D$ such that

$$\int_E Y d\mu = \int_E E[Y|X] d\mu \quad \text{for} \quad E \in \mathcal{F}(X).$$

It is well known that there exists a Borel measurable and μ_X-integrable function on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n), \mu_X)$ such that $E[Y|X] = \psi \circ X$, where μ_X is the probability measure defined by $\mu_X(B) = \mu(X^{-1}(B))$ for $B \in \mathcal{B}(\mathbb{R}^n)$. The function $\psi(\vec{\xi})$, $\vec{\xi} \in \mathbb{R}^n$ is unique up to Borel null sets in $\mathbb{R}^n$. Following Yeh [16] the function $\psi(\vec{\xi})$, written $E[Y|X = \vec{\xi}]$, is called the *conditional function space integral* of Y given X. Let W be a stochastic process on $(\mathbb{R}^D, \mathcal{B}^D, \mu)$ and D defined by $W(t, x) = x(t), t \in D, x \in \mathbb{R}^D$. Then W is a generalized Brownian motion process whose sample space is $\mathbb{R}^D$.

For each $t \in [0,T]$ and $\xi \in \mathbb{R}$, let Y_t and X_t be $\mathcal{B}^D$-measurable functions on $\mathbb{R}^D$ defined by

$$Y_t(x) = \exp\{\int_0^t \theta(s, x(s)+\xi)ds\} \quad \text{and} \quad X_t(x) = x(t)+\xi \tag{2.2}$$

where $\theta(\cdot,\cdot)$ is a complex valued Borel measurable function on $[0,T]\times\mathbb{R}$ for which Y_t is μ-integrable for each $(t,\xi) \in [0,T]\times\mathbb{R}$. In recent paper[5], it has been shown that the function U on $[0,T]\times\mathbb{R}\times\mathbb{R}$ defined by

$$U(t;\xi,\eta) = E[Y_t|X_t=\eta](2\pi b(t))^{-\frac{1}{2}}\exp\{-\frac{(\eta-a(t)-\xi)^2}{2b(t)}\}$$

satisfies the generalized Kac-Feynman integral equation

$$\begin{aligned} U(t;\xi,\eta) = {} & (2\pi b(t))^{-\frac{1}{2}}\exp\{-\frac{(\eta-a(t)-\xi)^2}{2b(t)}\} \\ & + \int_0^t\int_{\mathbb{R}} \theta(s,\zeta)U(s;\xi,\zeta)(2\pi(b(t)-b(s)))^{-\frac{1}{2}} \\ & \cdot \exp\{-\frac{((\zeta-a(s))-(\eta-a(t)))^2}{2(b(t)-b(s))}\}d\zeta ds. \end{aligned} \tag{2.3}$$

3. Existence of the partial differential equation

In this section, we will show that the function U given by (2.3) has the first and second partial derivatives with respect to η.

In order to do this, we shall require the following assumptions:

Assumption : We assume that $\theta(t,\eta)$ has the first partial derivative $\theta_\eta(t,\eta)$ such that

$$|\theta(t,\eta)| < K \quad \text{and} \quad |\theta_\eta(t,\eta)| < K$$

on $[0,T]\times\mathbb{R}$ for some constant $K>0$.

We also assume that $b(t)$ is continuously differentiable with $b'(t) \geq c_0$ for any $t\in[0,T]$ and $a(t)$ is continuous on $[0,T]$ for some constant $c_0>0$.

We start with the following lemma proved in [2](p.46, Corollary 5.9).

Lemma 3.1. *Suppose that for some $t_0 \in [a,b]$, the function $x \to f(x,t_0)$ is integrable on X, that $\partial f/\partial t$ exists on $X \times [a,b]$, and that there exists an integrable function g on X such that*

$$\left|\frac{\partial f}{\partial t}(x,t)\right| \le g(x).$$

Then the function

$$F(t) = \int f(x,t)d\mu(x)$$

is differentiable on $[a,b]$ and

$$\frac{\partial F}{\partial t}(t) = \frac{d}{dt}\int f(x,t)d\mu(x) = \int \frac{\partial f}{\partial t}(x,t)d\mu(x).$$

From Lemma 3.1, we obtain the following theorems.

Theorem 3.2. *Let $U(t;\xi,\eta)$ be as in (2.3). Then the function*

$$W(t;\xi,\eta) \equiv \int_0^t \int_{\mathbb{R}} \theta(s,\zeta)U(s;\xi,\zeta)(2\pi(b(t)-b(s)))^{-\frac{1}{2}} \cdot \exp\{-\frac{((\eta-a(t))-(\zeta-a(s)))^2}{2(b(t)-b(s))}\}d\zeta ds$$

has the first partial derivative with respect to η and

$$\frac{\partial W}{\partial \eta} = \int_0^t \int_{\mathbb{R}} \theta(s,\zeta)U(s;\xi,\zeta)(2\pi(b(t)-b(s)))^{-\frac{1}{2}} u \exp\{-\frac{u^2}{2}\}du ds$$

where $u = ((\zeta-a(s))-(\eta-a(t)))(b(t)-b(s))^{-\frac{1}{2}}$.

Proof. In view of Lemma 3.1, it suffices to show that

$$\int_0^t \int_{\mathbb{R}} \left|\frac{\partial}{\partial \eta}[\theta(s,\zeta)U(s;\xi,\zeta)(2\pi(b(t)-b(s))^{-\frac{1}{2}} \exp\{-\frac{((\eta-a(t))-(\zeta-a(s)))^2}{2(b(t)-b(s))}\}]\right| d\zeta ds \tag{3.1}$$

is finite. Since $\theta(s,\zeta)$ and $E[Y_s|x(s)+\xi=\zeta]$ are bounded, (3.1) is less than or equal to

$$K_1 \int_0^t \int_{\mathbb{R}} ((2\pi)^2 b(s)(b(t)-b(s)))^{-\frac{1}{2}} \left|\frac{(\eta-a(t))-(\zeta-a(s))}{b(t)-b(s)}\right| \cdot \exp\left\{-\frac{1}{2}\left(\frac{(\zeta-a(s)-\xi)^2}{b(s)} + \frac{((\eta-a(t))-(\zeta-a(s)))^2}{b(t)-b(s)}\right)\right\}d\zeta ds \tag{3.2}$$

for some constant $K_1 > 0$.
Observe that for any $s, t \in [0, T]$ with $s < t$,

$$\frac{v^2}{b(s)} + \frac{(u-v)^2}{b(t)-b(s)} = \frac{b(t)}{b(s)(b(t)-b(s))}\Big(v - \frac{b(s)}{b(t)}u\Big)^2 + \frac{u^2}{b(t)}\ .$$

Using this in (3.2), (3.2) is equal to

$$\begin{aligned} K_1 \int_0^t &\big((2\pi)^2 b(s)(b(t)-b(s))\big)^{-\frac{1}{2}} \exp\Big\{-\frac{(\eta - a(t) - \xi)^2}{2b(t)}\Big\} \\ &\cdot \int_{\mathbb{R}} \exp\Big\{-\frac{b(t)}{2b(s)(b(t)-b(s))}\big(\zeta - a(s) - \xi - \frac{b(s)}{b(t)}(\eta - a(t) - \xi)\big)^2\Big\} \\ &\qquad\qquad \cdot \Big|\frac{(\eta - a(t)) - (\zeta - a(s))}{b(t)-b(s)}\Big| d\zeta ds. \end{aligned} \tag{3.3}$$

We first consider the integral which appears in the above equation (3.3)

$$\begin{aligned} \int_{\mathbb{R}} \exp\Big\{-&\frac{b(t)}{2b(s)(b(t)-b(s))}[(\zeta - a(s) - \xi) \\ &- \frac{b(s)}{b(t)}(\eta - a(t) - \xi)]^2\Big\}\Big|\frac{(\zeta - a(s)) - (\eta - a(t))}{b(t)-b(s)}\Big| d\zeta. \end{aligned}$$

By the change of variable theorem, the above equation is equal to

$$\begin{aligned} &\int_{\mathbb{R}} \exp\Big\{-\frac{b(t)}{2b(s)(b(t)-b(s))}y^2\Big\}\Big|\frac{y + (\eta - a(t) - \xi)(\frac{b(s)}{b(t)} - 1)}{b(t)-b(s)}\Big| dy \\ &\le \int_{\mathbb{R}} \exp\Big\{-\frac{b(t)}{2b(s)(b(t)-b(s))}y^2\Big\}\frac{|y|}{b(t)-b(s)} dy \\ &\qquad + \int_{\mathbb{R}} \exp\Big\{-\frac{b(t)}{2b(s)(b(t)-b(s))}y^2\Big\}\Big|\frac{\eta - a(t) - \xi}{b(t)}\Big| dy \\ &= 2\frac{b(s)}{b(t)} + \frac{|\eta - a(t) - \xi|}{b(t)}\Big(\frac{2\pi b(s)(b(t)-b(s))}{b(t)}\Big)^{\frac{1}{2}} \end{aligned}$$

where $y = \zeta - a(s) - \xi - (\eta - a(t) - \xi)b(s)/b(t)$. Using this in (3.3), (3.3) is less than or equal to

$$\begin{aligned} K_1 \int_0^t &\frac{1}{\sqrt{(2\pi)^2 b(s)(b(t)-b(s))}} \exp\Big\{-\frac{(\eta - a(t) - \xi)^2}{2b(t)}\Big\} \\ &\cdot \Big[\frac{2b(s)}{b(t)} + \frac{|\eta - a(t) - \xi|}{b(t)}\Big(\frac{2\pi b(s)(b(t)-b(s))}{b(t)}\Big)^{\frac{1}{2}}\Big] ds. \end{aligned} \tag{3.4}$$

But by the change of variable theorem, we have

$$\begin{aligned}
&\int_0^t \frac{1}{\sqrt{(2\pi)^2 b(s)(b(t)-b(s))}} \times \frac{2b(s)}{b(t)} ds = \int_0^t \frac{1}{\sqrt{\pi^2(b(t)-b(s))}} \times \frac{\sqrt{b(s)}}{b(t)} ds \\
&\leq \frac{1}{\pi\sqrt{b(t)}} \int_0^t \frac{1}{\sqrt{b(t)-b(s)}} ds \leq \frac{1}{\pi\sqrt{b(t)}} \cdot \frac{1}{C_0} \int_0^{b(t)} \frac{1}{\sqrt{u}} du = \frac{2}{\pi C_0} < \infty
\end{aligned}$$

since $b(s)/b(t) \leq 1$ for every $0 \leq s < t \leq T$ and $b'(t) \geq c_0$ for any $t \in [0,T]$. Hence using this in (3.4), (3.4) is less than or equal to

$$\frac{2}{\pi b'(0)} + \frac{t}{\sqrt{2\pi b(t)}} \left(\frac{|\eta - a(t) - \xi|}{b(t)} \right)^{\frac{1}{2}} < \infty.$$

Thus $W(t;\xi,\eta)$ has the first partial derivative with respect to η and

$$\begin{aligned}
\frac{\partial W}{\partial \eta} = \int_0^t \int_{\mathbb{R}} &\theta(s,\zeta) U(s;\xi,\zeta)(2\pi(b(t)-b(s)))^{-\frac{1}{2}} \\
&\cdot \exp\left\{ -\frac{((\eta - a(t)) - (\zeta - a(s)))^2}{2(b(t)-b(s))} \right\} \left(-\frac{((\eta - a(t)) - (\zeta - a(s)))}{b(t)-b(s)} \right) d\zeta ds.
\end{aligned}$$

Theorem 3.3. *Let $W(t;\xi,\eta)$ be as in Theorem 3.2. Then the function $W(t;\xi,\eta)$ has the second partial derivative with respect to η.*

Proof. By Theorem 3.2, $\dfrac{\partial W}{\partial \eta}$ exists and is given in Theorem 3.2. Then we have

$$\frac{\partial W}{\partial \eta} = \int_0^t \int_{\mathbb{R}} \theta(s,\zeta) U(s;\xi,\zeta)(2\pi(b(t)-b(s)))^{-\frac{1}{2}} u \exp\{-\frac{u^2}{2}\} du ds.$$

where u is as in Theorem 3.2.

In view of Lemma 3.1, it suffices to show that

$$\int_0^t \int_{\mathbb{R}} \left| \frac{\partial}{\partial \eta} [\theta(s,\zeta) U(s;\xi,\zeta)] (2\pi(b(t)-b(s)))^{-\frac{1}{2}} u \exp\{-\frac{u^2}{2}\} \right| du ds \tag{3.5}$$

is finite. But (3.5) is equal to

$$\int_0^t \int_{\mathbb{R}} |\theta_\zeta(s,\zeta) U(s;\xi,\zeta) + \theta(s,\zeta) U_\zeta(s;\xi,\zeta)| (2\pi(b(t)-b(s)))^{-\frac{1}{2}} |u| \exp\{-\frac{u^2}{2}\} du ds.$$

By assumption, $\theta_\zeta(s,\zeta)$ is bounded and hence we can see that

$$\int_0^t \int_{\mathbb{R}} |\theta_\zeta(s,\zeta) U(s;\xi,\zeta)| (2\pi(b(t)-b(s)))^{-\frac{1}{2}} |u| \exp\{-\frac{u^2}{2}\} du ds$$

is finite. Hence it remains to show that

$$\int_0^t \int_{\mathbb{R}} \big|\theta(s,\zeta)U_\zeta(s;\xi,\zeta)\big|(2\pi(b(t)-b(s)))^{-\frac{1}{2}}|u|\exp\{-\frac{u^2}{2}\}duds \tag{3.6}$$

is finite. By Theorem 3.2, $\partial W/\partial\eta$ exists and so $\partial U/\partial\eta$ exists. Moreover, by the definition of $U(s;\xi,\zeta)$, we can see that

$$\begin{aligned}
&U(s;\xi,\zeta)\\
&= E\big[\exp\{\int_0^s \theta(w,x(w)+\xi)dw\}|x(s)+\xi=\zeta\big](2\pi b(s))^{-\frac{1}{2}}\exp\Big\{-\frac{(\zeta-a(s)-\xi)^2}{2b(s)}\Big\}\\
&= (2\pi b(s))^{-\frac{1}{2}}\exp\Big\{-\frac{(\eta-a(t)+\sqrt{b(t)-b(s)}u-\xi)^2}{2b(s)}\Big\}\\
&\quad\cdot\int_{\mathbb{R}^D}\exp\Big\{\int_0^s \theta\big(w,x(w)-\frac{b(w)}{b(s)}x(s)+\frac{b(w)}{b(s)}(\eta+a(s)-a(t)\\
&\qquad\qquad +\sqrt{b(t)-b(s)}u-\xi)+\xi\big)dw\Big\}d\mu(x)
\end{aligned}$$

Thus we have

$$\begin{aligned}
&\frac{\partial U}{\partial\zeta}(s;\xi,\zeta) = (2\pi b(s))^{-\frac{1}{2}}\exp\Big\{-\frac{(\eta-a(t)+\sqrt{b(t)-b(s)}u-\xi)^2}{2b(s)}\Big\}\\
&\quad\cdot\Big[\int_0^s \frac{\partial}{\partial\zeta}\theta\big(w,x(w)-\frac{b(w)}{b(s)}x(s)+\frac{b(w)}{b(s)}(\eta+a(s)-a(t)\\
&\qquad\qquad +\sqrt{b(t)-b(s)}u-\xi)-\xi\big)dw\\
&\qquad\qquad -\frac{(\eta-a(t)+\sqrt{b(t)-b(s)}u-\xi)}{b(s)}\Big]\\
&\quad\cdot\int_{\mathbb{R}^D}\exp\Big\{\int_0^s \theta\big(w,x(w)-\frac{b(w)}{b(s)}x(s)\\
&\qquad\qquad +\frac{b(w)}{b(s)}(\eta+a(s)-a(t)+\sqrt{b(t)-b(s)}u-\xi)+\xi\big)dw\Big\}d\mu(x)
\end{aligned}$$

Using this in (3.6), (3.6) is less than or equal to

$$
\begin{aligned}
&\int_0^t \int_{\mathbb{R}} ((2\pi)^2 b(s)(b(t)-b(s)))^{-\frac{1}{2}} \Bigg| \theta(s,\zeta) \\
&\quad \cdot \exp\left\{ -\frac{(\eta - a(t) + \sqrt{b(t)-b(s)}u - \xi)^2}{2b(s)} \right\} |u| \exp\{-\frac{u^2}{2}\} \\
&\quad \cdot \left| \frac{b(w)}{b(s)} \left(\int_0^s \theta_\zeta \left(w, x(w) - \frac{b(w)}{b(s)} x(s) + \frac{b(w)}{b(s)} (\eta + a(s) - a(t) + \sqrt{b(t)-b(s)}u - \xi) \right.\right.\right. \\
&\qquad\qquad \left.\left. + \xi \right) dw \right) + \frac{\eta - a(t) + \sqrt{b(t)-b(s)}u - \xi}{b(s)} \Bigg| duds \\
&\leq K_2 \left[\int_0^t \int_{\mathbb{R}} ((2\pi)^2 b(s)(b(t)-b(s)))^{-\frac{1}{2}} |u| \exp\{-\frac{u^2}{2}\} duds \right. \\
&\qquad + \int_0^t \int_{\mathbb{R}} ((2\pi)^2 b(s)(b(t)-b(s)))^{-\frac{1}{2}} \left| \frac{\eta - a(t) + \sqrt{b(t)-b(s)}u - \xi}{b(s)} \right| \\
&\qquad\qquad \left. \cdot \exp\left\{ -\frac{(\eta - a(t) + \sqrt{b(t)-b(s)}u - \xi)^2}{2b(s)} - \frac{u^2}{2} \right\} |u| duds \right]
\end{aligned}
\tag{3.7}
$$

for some constant $K_2 > 0$ since $\theta(s,\zeta)$ and $\partial\theta(s,\zeta)/\partial\zeta$ are bounded and $|b(w)/b(s)| \leq 1$ for every $0 \leq w < s \leq T$. But we have that

$$
\begin{aligned}
&-\frac{(\eta - a(t) - \xi + \sqrt{b(t)-b(s)}u)^2}{2b(s)} - \frac{u^2}{2} \\
&\quad = -\frac{b(t)\left(u + (\eta - a(t) - \xi)\sqrt{b(t)-b(s)}/b(t)\right)^2}{2b(s)} - \frac{(\eta - a(t) - \xi)^2}{2b(t)}.
\end{aligned}
$$

Using this, we consider the last integral which appears in the right-hand side of (3.7)

$$
\begin{aligned}
&\int_0^t \int_{\mathbb{R}} ((2\pi)^2 b(s)(b(t)-b(s)))^{-\frac{1}{2}} \left| \frac{\eta - a(t) + \sqrt{b(t)-b(s)}u - \xi}{b(s)} \right| \\
&\qquad \cdot \exp\left\{ -\frac{(\eta - a(t) + \sqrt{b(t)-b(s)}u - \xi)^2}{2b(s)} - \frac{u^2}{2} \right\} |u| duds \\
&= \int_0^t \int_{\mathbb{R}} ((2\pi)^2 b(s)(b(t)-b(s)))^{-\frac{1}{2}} \exp\left\{ -\frac{b(t)}{2b(s)} \left(u + \frac{\sqrt{b(t)-b(s)}}{b(t)} (\eta - a(t) - \xi) \right)^2 \right\} \\
&\qquad \cdot \exp\left\{ -\frac{(\eta - a(t) - \xi)^2}{2b(t)} \right\} \left| \frac{\eta - a(t) + \sqrt{b(t)-b(s)}u - \xi}{b(s)} \right| |u| duds.
\end{aligned}
$$

Let $v = u + ((\eta - a(t) - \xi)\sqrt{b(t) - b(s)})/b(t)$. Then by the change of variable theorem, the above equation is equal to

$$\int_0^t \big((2\pi)b(s)(b(t) - b(s))\big)^{-\frac{1}{2}} \exp\{-\frac{(\eta - a(t) - \xi)^2}{2b(t)}\}$$
$$\cdot \int_{\mathbb{R}} \left| v - \frac{\sqrt{b(t) - b(s)}}{b(t)}(\eta - a(t) - \xi)\right| \cdot \left|\frac{(\eta - a(t) - \xi)}{b(t)} + \frac{\sqrt{b(t) - b(s)}}{b(s)} v\right|$$
$$\exp\{-\frac{b(t)}{2b(s)}v^2\}dvds$$
$$= \int_0^t \big((2\pi b(s)(b(t) - b(s))\big)^{-\frac{1}{2}} \exp\{-\frac{(\eta - a(t) - \xi)^2}{2b(t)}\}$$
$$\cdot \int_{\mathbb{R}} \left| \sqrt{\frac{b(s)}{b(t)}} y - \frac{\sqrt{b(t) - b(s)}}{b(t)}(\eta - a(t) - \xi)\right| \sqrt{\frac{b(s)}{b(t)}}$$
$$\cdot \left|\frac{(\eta - a(t) - \xi)}{b(t)} + \frac{\sqrt{b(t) - b(s)}}{b(s)} \times \sqrt{\frac{b(s)}{b(t)}} y\right| \{-\frac{y^2}{2}\}dyds$$

$$\leq \int_0^t \big((2\pi)^2 b(s)(b(t) - b(s))\big)^{-\frac{1}{2}}$$
$$\cdot \Big[\int_{\mathbb{R}} \frac{\sqrt{b(t) - b(s)}}{b(t)} y^2 \exp\{-\frac{y^2}{2}\}dy + \frac{2}{b(t)}|\eta - a(t) - \xi| \int_{\mathbb{R}} |y| \exp\{-\frac{y^2}{2}\}dy$$
$$+ \frac{(\eta - a(t) - \xi)^2\sqrt{b(t)}}{b(t)^2} \int_{\mathbb{R}} \exp\{-\frac{y^2}{2}\}dy\Big] ds$$
$$\leq \frac{1}{C_0}\left[\frac{\sqrt{2}}{\sqrt{\pi b(t)}} + \frac{4\pi(\eta - a(t) - \xi)}{b(t)} + \frac{\pi\sqrt{2\pi b(t)}(\eta - a(t) - \xi)^2}{b(t)^2}\right]$$

where $y = \sqrt{\frac{b(t)}{b(s)}}v$ and we have used $\exp\{-(\eta - a(t) - \xi)^2/2b(t)\} \leq 1$ and $b'(t) \geq c_0$ for every $t \in [0, T]$. By using this in (3.7), (3.7) is finite. Thus $W(t; \xi, \eta)$ has the second partial derivative with respect to η.

4. The partial differential equation

In this section we will show that the generalized Kac-Feynman integral equation (2.3) is equivalent to a partial differential equation which is generalized form of the equation (1.3). That is, the function U given by (2.3) satisfies the patial differential equation

$$\frac{\partial U}{\partial t} = \frac{1}{2}b'(t)\frac{\partial^2 U}{\partial \eta^2} - a'(t)\frac{\partial U}{\partial \eta} + \theta(t,\eta)U(t;\xi,\eta) \tag{4.1}$$

with the initial condition $\lim_{t\to 0^+} U(t;\xi,\eta) = \delta(\eta-\xi)$.

Main theorem. For each $t \in [0,T]$ and $\xi \in \mathbb{R}$, let X_t and Y_t be as in (2.2). Then the function U given by (2.3) satisfies the partial differential equation (4.1).

Proof. Let $W(t;\xi,\eta)$ be as in Theorem 3.2. Then by Theorem 3.2 and 3.3, $\frac{\partial W}{\partial \eta}$ and $\frac{\partial^2 W}{\partial \eta^2}$ exist. Let u be as in Theorem 3.2. Then

$$\frac{\partial W}{\partial \eta} = \int_0^t \int_{\mathbb{R}} \theta(s,\zeta)U(s;\xi,\zeta)(2\pi(b(t)-b(s)))^{-\frac{1}{2}} u \exp\{-\frac{u^2}{2}\}duds$$

and

$$\frac{\partial^2 W}{\partial \eta^2} = \int_0^t \int_{\mathbb{R}} \frac{\partial}{\partial \eta}[\theta(s,\zeta)U(s;\xi,\zeta)](2\pi(b(t)-b(s)))^{-\frac{1}{2}} u \exp\{-\frac{u^2}{2}\}duds.$$

Thus by using these we see that

$$\begin{aligned}\frac{\partial U}{\partial \eta} &= (2\pi b(t))^{-\frac{1}{2}} \exp\{-\frac{(\eta-a(t)-\xi)^2}{2b(t)}\}(-\frac{\eta-a(t)-\xi}{b(t)}) \\ &\quad + \int_0^t \int_{\mathbb{R}} \theta(s,\zeta)U(s;\xi,\zeta)(2\pi(b(t)-b(s)))^{-\frac{1}{2}} u \exp\{-\frac{u^2}{2}\}duds\end{aligned}$$

and

$$\begin{aligned}\frac{\partial^2 U}{\partial \eta^2} &= (2\pi b(t))^{-\frac{1}{2}} \exp\{-\frac{(\eta-a(t)-\xi)^2}{2b(t)}\}(\frac{\eta-a(t)-\xi}{b(t)})^2 \\ &\quad + (2\pi b(t))^{-\frac{1}{2}} \exp\{-\frac{(\eta-a(t)-\xi)^2}{2b(t)}\}(-\frac{1}{b(t)}) \\ &\quad + \int_0^t \int_{\mathbb{R}} \frac{\partial}{\partial \eta}[\theta(s,\zeta)U(s;\xi,\zeta)](2\pi(b(t)-b(s)))^{-\frac{1}{2}} u \exp\{-\frac{u^2}{2}\}duds.\end{aligned}$$

Note that

$$\frac{\partial}{\partial \eta}[\theta(s,\zeta)U(s;\xi,\zeta)] = \theta_\zeta(s;\zeta)U(s;\xi,\zeta) + \theta(s;\zeta)U_\zeta(s;\xi,\zeta)$$

and

$$\begin{aligned}\frac{\partial}{\partial t}[\theta(s,\zeta)U(s;\xi,\zeta)] &= -a'(t)\big(\theta_\zeta(s,\zeta)U(s;\xi,\zeta)+\theta(s,\zeta)U_\zeta(s;\xi,\zeta)\big)\\ &+\frac{1}{2}(b(t)-b(s))^{-\frac{1}{2}}\cdot u\cdot b'(t)\big(\theta_\zeta(s,\zeta)U(s;\xi,\zeta)+\theta(s,\zeta)U_\zeta(s;\xi,\zeta)\big).\end{aligned} \tag{4.2}$$

Hence we obtain

$$\begin{aligned}&\frac{1}{2}\cdot u\cdot b'(t)(b(t)-b(s))^{-\frac{1}{2}}\big(\theta_\zeta(s,\zeta)U(s;\xi,\zeta)+\theta(s,\zeta)U_\zeta(s;\xi,\zeta)\big)\\ &=\frac{\partial}{\partial t}[\theta(s,\zeta)U(s;\xi,\zeta)]+a'(t)\big(\theta_\zeta(s,\zeta)U(s;\xi,\zeta)+\theta(s,\zeta)U_\zeta(s;\xi,\zeta)\big).\end{aligned} \tag{4.3}$$

By (4.2) and (4.3), we have

$$\begin{aligned}\frac{1}{2}b'(t)\frac{\partial^2 U}{\partial\eta^2}(s;\xi,\zeta) &= (2\pi b(t))^{-\frac{1}{2}}\exp\{-\frac{(\eta-a(t)-\xi)^2}{2b(t)}\}(-\frac{\eta-a(t)-\xi}{b(t)})^2(\frac{1}{2}b'(t))\\ &+(2\pi b(t))^{-\frac{1}{2}}\exp\{-\frac{(\eta-a(t)-\xi)^2}{2b(t)}\}(-\frac{1}{b(t)})(\frac{1}{2}b'(t))\\ &+\int_0^t\int_{\mathbb{R}}\frac{\partial}{\partial t}[\theta(s,\zeta)U(s;\xi,\zeta)](2\pi)^{-\frac{1}{2}}\exp\{-\frac{u^2}{2}\}duds\\ &+a'(t)\frac{\partial}{\partial\eta}\int_0^t\int_{\mathbb{R}}[\theta(s,\zeta)U(s;\xi,\zeta)](2\pi)^{-\frac{1}{2}}\exp\{-\frac{u^2}{2}\}duds.\end{aligned}$$

Now, consider the integral

$$\int_0^t\int_{\mathbb{R}}\frac{\partial}{\partial t}[\theta(s,\zeta)U(s;\xi,\zeta)](2\pi)^{-\frac{1}{2}}\exp\{-\frac{u^2}{2}\}duds. \tag{4.4}$$

By integrating the both sides of (4.4) with respect to t from 0 to v, we obtain

$$\begin{aligned}&\int_0^v\int_0^t\int_{\mathbb{R}}\frac{\partial}{\partial t}[\theta(s,\zeta)U(s;\xi,\zeta)](2\pi)^{-\frac{1}{2}}\exp\{-\frac{u^2}{2}\}dudsdt\\ &=\int_0^v\int_{\mathbb{R}}\int_s^v\frac{\partial}{\partial t}[\theta(s,\zeta)U(s;\xi,\zeta)](2\pi)^{-\frac{1}{2}}\exp\{-\frac{u^2}{2}\}dtduds\\ &=\int_0^v\int_{\mathbb{R}}[\theta(s,\zeta)U(s;\xi,\zeta)]_s^v(2\pi)^{-\frac{1}{2}}\exp\{-\frac{u^2}{2}\}duds\\ &=\int_0^v\int_{\mathbb{R}}\theta(s,\eta+a(s)-a(v)+\sqrt{b(v)-b(s)}u)(2\pi)^{-\frac{1}{2}}\exp\{-\frac{u^2}{2}\}\\ &\qquad\cdot U(s;\xi,\eta+a(s)-a(v)+\sqrt{b(v)-b(s)}u)duds\ -\ \int_0^v\theta(s,\eta)U(s;\xi,\eta)ds.\end{aligned}$$

So, by differntiating with respect to v and replacing v by t, we have

$$\int_0^t \int_{\mathbb{R}} \frac{\partial}{\partial t}[\theta(s,\zeta)U(s;\xi,\zeta)](2\pi)^{-\frac{1}{2}} \exp\{-\frac{u^2}{2}\}duds \tag{4.5}$$
$$= \frac{\partial}{\partial t}\int_0^t \int_{\mathbb{R}} [\theta(s,\zeta)U(s;\xi,\zeta)](2\pi)^{-\frac{1}{2}} \exp\{-\frac{u^2}{2}\}duds \ - \ \theta(t,\eta)U(t;\xi,\eta).$$

Thus we have

$$\begin{aligned}\frac{1}{2}b'(t)\frac{\partial^2 U}{\partial \eta^2} &= (2\pi b(t))^{-\frac{1}{2}} \exp\{-\frac{(\eta - a(t) - \xi)^2}{2b(t)}\}(\frac{1}{2}b'(t))(\frac{\eta - a(t) - \xi}{b(t)})^2 \\ &+ (2\pi b(t))^{-\frac{1}{2}} \exp\{-\frac{(\eta - a(t) - \xi)^2}{2b(t)}\}(-\frac{1}{b(t)})(\frac{1}{2}b'(t)) \\ &+ \frac{\partial}{\partial t}\int_0^t \int_{\mathbb{R}} [\theta(s,\zeta)U(s;\xi,\zeta)](2\pi)^{-\frac{1}{2}} \exp\{-\frac{u^2}{2}\}duds \ - \ \theta(t,\eta)U(t;\xi,\eta) \\ &+ a'(t)\frac{\partial}{\partial \eta}\int_0^t \int_{\mathbb{R}} [\theta(s,\zeta)U(s;\xi,\zeta)](2\pi)^{-\frac{1}{2}} \exp\{-\frac{u^2}{2}\}duds.\end{aligned}$$

But

$$\frac{\partial}{\partial t}[(2\pi b(t))^{-\frac{1}{2}} \exp\{-\frac{(\eta - a(t) - \xi)^2}{2b(t)}\}] \tag{4.6}$$
$$\begin{aligned}&= (2\pi b(t))^{-\frac{1}{2}} \exp\{-\frac{(\eta - a(t) - \xi)^2}{2b(t)}\}(-\frac{1}{2}b'(t))(\frac{1}{b(t)}) \\ &+ (2\pi b(t))^{-\frac{1}{2}} \exp\{-\frac{(\eta - a(t) - \xi)^2}{2b(t)}\}(\frac{\eta - a(t) - \xi}{b(t)})a'(t) \\ &+ (2\pi b(t))^{-\frac{1}{2}} \exp\{-\frac{(\eta - a(t) - \xi)^2}{2b(t)}\}(\frac{\eta - a(t) - \xi}{b(t)})^2(\frac{1}{2}b'(t)).\end{aligned}$$

By using (4.6), we have

$$\frac{\partial U}{\partial t} = \frac{1}{2}b'(t)\frac{\partial^2 U}{\partial \eta^2} - a'(t)\frac{\partial U}{\partial \eta} + \theta(t,\eta)U(t;\xi,\eta)$$

which completes the proof of the theorem.

Remark. In Main Theorem, if $\{x(t) : t \in [0,T]\}$ is the standard Wiener process, then $a(t) \equiv 0$ and $b(t) = t$ and hence the function $U(t;\xi,\eta)$ satisfies the following partial differential equation

$$\frac{\partial U}{\partial t} = \frac{1}{2}\frac{\partial^2 U}{\partial \eta^2} + \theta(t,\eta)U(t;\xi,\eta).$$

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DEPARTMENT OF MATHEMATICS, DANKOOK UNIVERSITY, CHEONAN 330-714, KOREA