Fuzzy Syntopogenous Structures and Orders

Sehwa Chung*, Moohong Joo*

※This research was supported by a grant from Kyungwon University.

ABSTRACT

We introduce the category [PFSyn] of saturated fuzzy syntopogenous preordered spaces and continuous isotones and show that the category [PFSyn] is topological and cotopological. Furthermore, to consider a compatibility between order structures and fuzzy syntopogenous structures, we introduce a category [IPFSyn] of increasing saturated fuzzy syntopogenous spaces and its dual category [IPFSyn] of decreasing saturated fuzzy syntopogenous spaces, and show that [IPFSyn] and [DPFSyn] are both bireflective in the category [PFSyn].

I. Introduction

In order to describe the nearness between fuzzy sets various fuzzy structures like the fuzzy neighborhood structure ([L1]), the Artico-Moresco saturated fuzzy proximity ([C1]) and the Lowen fuzzy uniformity ([L2]) have been introduced. The concept of saturated fuzzy syntopogenous structures has been introduced by Chung in [C1] to generalize the above three structures.

In [AHE] Allam, Hussein and El-Saady introduced an order in fuzzy syntopogenous spaces. But their definition is not clear since $x \leq_S y$ does not imply $y \leq_{S^c} x$ in general for given a fuzzy syntopogenous structure S on a set X.

In this paper to clarify the above definition we introduce a new order in fuzzy syntopogenous space and investigate its several properties including the fact that the category [PFSyn] of saturated fuzzy syntopogenous preordered spaces is topological and cotopological. Moreover, we show that the category [IPFSyn] of increasing saturated fuzzy syntopogenous spaces and its dual category [DPFSyn] of decreasing saturated fuzzy syntopogenous spaces are both bireflective in the category [PFSyn].

We will follow the usual notation convention in the fuzzy set theory: The unit in terval is denoted by I. If $A \subseteq X$, we use also A to denote the fuzzy set which is equal to the characteristic map of A and x the fuzzy set $\{x\}$. For any $a \in I$, the constant map on X into I with the value a is also denoted by a. For $A \subseteq X$ and $a \in I$, $a \wedge A$ and $a \vee A$ will be denoted by A_a and A^a, respectively.

For the details of fuzzy syntopogenous structures we refer to [C1, C2, K] and for the category theory we refer to [AHS]. However, we recall the following definitions: A *fuzzy syntopogenous structure* on a set X is a family S of fuzzy semi-topogenous orders on X satisfies the following:

(FS1) S is directed, that is, given $\tau, \tau' \in S$ there exists

*Department of Mathematics, Kyungwon University

$\eta \in S$ with $\tau, \tau' \leq \eta$.

(FS2) Given $\tau \in S$ and $\varepsilon > 0$ there exists $\eta \in S$ such that $\tau \leq \eta \circ \eta + \varepsilon$.

A fuzzy syntopogenous structure S on a set X is said to be *saturated* if every member of S is saturated. A map f of a fuzzy syntopogenous space (X, S) into another one (Y, S') is said to be continuous if for each $\tau \in S'$ and $\varepsilon > 0$ there exists $\eta \in S$ such that $f^{-1}(\tau) \leq \eta + \varepsilon$, where $f^{-1}(\tau)(\alpha, \beta) = \tau(f(\alpha), 1 - f(1-\beta))(\alpha, \beta \in I^X)$. The category of all saturated fuzzy syntopogenous spaces and continuous maps is denoted by [FSyn].

In addition we use the following notation and terminology relating to orderings. A preorder R on a set X is a reflexive and transitive relation on X and the pair (X, R) is called a *preordered set*. A map f of the preordered set (X, R) into another one (Y, R') is said to be an *isotone* if $(x, y) \in R$ implies $(f(x), f(y)) \in R'$. Let R and R' be preorders on X. Then we say that R is finer than R' if the identity map $id_X : (X, R') \to (X, R)$ is an isotone. In this case we write $R' \subseteq R$. The category of all preordered sets and isotones is denoted by [Pord]. Finally note that throughout this paper, unless otherwise stated, (X, R) is a preordered set.

II. Fuzzy Syntopogenous Preordered Spaces

A *fuzzy syntopogenous preordered space* is a triplet (X, S, R) consisting of a set X, a fuzzy syntopogenous structure S and a preorder R on X.

The category of all saturated fuzzy syntopogenous preordered spaces and isotones is denoted by [PFSyn].

Remark. (1) Every saturated fuzzy syntopogenous space (X, S) can be regarded as a saturated fuzzy syntopogenous preordered space $(X, S, =)$, where $=$ is the discrete order on X, and a map $f : (X, S) \to (Y, S')$ is continuous if and only if $f : (X, S, =) \to (Y, S', =)$ is a continuous isotone. Hence [FSyn] can be considered as a full subcategory of [PFSyn]. Futhermore, for any $(X, S, R) \in$ [PFSyn] the identity map $id_X : (X, S, =) \to (X, S, R)$ is a continuous isotone which is, in fact, the [FSyn]-coreflection of (X, S, R). Hence [PFSyn] contains [FSyn] as a coreflective subcategory.

(2) Every preordered set (X, R) can be regarded as a saturated fuzzy syntopogenous preordered space (X, D, R), where D is the discrete saturated fuzzy syntopogenous structure on X, and a map $f : (X, R) \to (Y, R')$ is as isotone if and only if $f : (X, D, R) \to (Y, D, R')$ is a continuous isotone. Thus [Pord] is also considered as a full subcategory which is coreflective in the category [PFSyn].

The following is immediate from the fact that [FSyn] and [Pord] are both topological.

Theorem 2.1. *The category* [PFSyn] *is topological and cotopological.*

For an ordinary operation ^{o} we denote by [^{o}-PFSyn] the full subcategory of [PFSyn] consisting of those fuzzy syntopogenous spaces (X, S, R) with $S = S^o$ (cf. [C2]).

Theorem 2.2. *If* ^{o} *and* ^{k} *are ordinary operations such that* $A^o \leq A^k$ *holds for any order family* A *then* [^{k}-PFSyn] *is coreflective in the category* [^{o}-PFSyn].

Proof. It is immediate from Proposition 3.7 in [C2].

III. Fuzzy Syntopogenous Structures Compatible with Orders

Let τ be a fuzzy semi-topogenous order on a set X and let R_τ be defined by

$$(x, y) \in R_\tau \text{ if and only if } \tau(x_a, z^b) \leq \tau(y_a, z^b) \text{ and } \tau(z_b, y^a) \leq \tau(z_b, x^a)$$

for $x, y, z \in X$ and $a, b, \in I$. Then R_τ is a preorder on X which is called the *preorder generated by the fuzzy semi-topogenous order* τ.

Remark. If τ is a symmetric fuzzy semi-topogenous order on a set X then R_τ is an equivalence relation on X.

Let X be a set and $B \subseteq X$. Then it is easy to show that for any $a \in I$

$$B_a = \vee_{x \in B} x_a \text{ and } B^a = \wedge_{x \in B} x^a.$$

Using this fact we have the following:

Proposition 3.1. *If τ is a saturated fuzzy topogenous order on a set X then*

$$(x, y) \in R_\tau \text{ if and only if } \tau(\{x, y\}_a, \alpha) = \tau(x_a, \alpha) \text{ and } \tau(\alpha, \{x, y\}^a) = \tau(\alpha, y^a)$$

for $x, y, z \in X$ *and* $a, b \in I$.

Proof. Immediate from the fact that every saturated fuzzy semi-topogenous order is topogenous.

Definition 3.2. A fuzzy semi-topogenous order τ on (X, R) is *increasing* (*decreasing*, resp.) if $R \subseteq R_\tau$ ($R \subseteq R_{\tau^c}$, resp.)

Notation. *Let X be a set and A a subset of X. Then we write*

(1) $\uparrow A = \{y \in X : (x, y) \in R$ for *some* $x \in A\}$.

(2) $\downarrow A = \{y \in X : (y, x) \in R$ for *some* $x \in A\}$.

Recall that if τ is saturated then for a family $\{A_i\}_{i \in I}$ of subsets of X and $\alpha \in I^X$ we have $\tau(\cup_{i \in I} A_i, \alpha) = \wedge_{i \in I} \tau(A_i, \alpha)$.

Using this fact, one has the following:

Proposition 3.3. *Let τ be a fuzzy semi-topogenous order on (X, R). Then one has the following:*

(1) *If τ is saturated and increasing then for $\alpha, \beta \in I^X$ and $x \in X$:*

(i) $\tau(x, \alpha) = \tau(\uparrow x, \alpha)$.

(ii) $\tau \circ \tau(\alpha, \beta) = \vee\{\tau(\alpha, A) \wedge \tau(A, \beta) : A = \uparrow A\}$

(2) *If τ is saturated and decreasing then for $\alpha, \beta \in I^X$ and $x \in X$:*

(i) $\tau(x, \alpha) = \tau(\downarrow x, \alpha)$.

(ii) $\tau \circ \tau(\alpha, \beta) = \vee\{\tau(\alpha, A) \wedge \tau(A, \beta) : A = \downarrow A\}$

(3) *If $\tau(\alpha, \beta) = \vee\{\tau(\alpha, A) \wedge \tau(A, \beta) : A = \uparrow A\}$ for $\alpha, \beta \in I^X$ and $(x, y) \in R$, then $\tau(x, z^b) \leq \tau(y, z^b)$.*

(4) *If $\tau(\alpha, \beta) = \vee\{\tau(\alpha, A) \wedge \tau(A, \beta) : A = \downarrow A\}$ for $\alpha, \beta \in I^X$ and $(y, x) \in R$, then $\tau(x, z^b) \leq \tau(y, z^b)$.*

Proof. (1) (i) Take any $y \in \uparrow x$. Since τ is increasing and saturated, $\tau(x, \alpha) \leq \tau(y, \alpha)$. Since R is reflexive we have $x \in \uparrow x$ and hence $\tau(x, \alpha) = \wedge_{y \in \uparrow x} \tau(y, \alpha)$. Thus, by the above fact, $\tau(x, \alpha) = \tau(\uparrow x, \alpha)$.

(ii) It is immediate from the definition of $\tau \circ \tau$ that $\tau \circ \tau(\alpha, \beta)$ is an upper bound of $\{\tau(\alpha, A) \wedge \tau(A, \beta) : A = \uparrow A\}$. Suppose v is an upper bound of $\{\tau(\alpha, A) \wedge \tau(A, \beta) : A = \uparrow A\}$. By the above fact and (i), for $A \subseteq X$, $\tau(A, \beta) = \tau(\uparrow A, \beta)$ and so $\tau(\alpha, A) \wedge \tau(A, \beta) \leq \tau(\alpha, \uparrow A) \wedge \tau(\uparrow A, \beta)$. Thus v is an upper bound of $\{\tau(\alpha, A) \wedge \tau(A, \beta) : A \subseteq X\}$ and hence $\tau \circ \tau(\alpha, \beta) \leq v$. Therefore, $\tau \circ \tau(\alpha, \beta) = \vee\{\tau(\alpha, A) \wedge \tau(A, \beta) : A = \uparrow A\}$.

(2) It is similar to (1).

(3) Suppose that $\tau(x, z^b) > \tau(y, z^b)$ for some $z \in X$ and $a \in I$. Since $\tau(x, z^b) = \vee\{\tau(x, A) \wedge \tau(A, z^b) : A = \uparrow A\}$, there is a subset of X such that $\tau(y, z^b) < \tau(x, A) \wedge \tau(A, z^b)$ and $A = \uparrow A$. If $x \in A$ then $y \in A$ and so $\tau(x, A) \wedge \tau(A, z^b) \leq \tau(A, z^b) \leq \tau(y, z^b)$, which contradict $\tau(y, z^b) < \tau(x, A) \wedge \tau(A, z^b)$. If $x \notin A$ then $\tau(x, A) = 0$. Thus $\tau(y, z^b) < 0$, which contradict $0 \leq \tau(y, z^b)$. Therefore, $\tau(x, z^b) \leq \tau(y, z^b)$ for all $z \in X$ and $b \in I$.

(4) It is similar to (3).

In the remainder of the section, (X, S, R) always denotes a fuzzy syntopogenous preordered space.

Definition 3.4. A space (X, S, R) is said to be *increasing* (*decreasing*, resp.) if every member of S is increasing (decreasing, resp.).

Remark. Since $\tau = \tau^{cc}$ for any fuzzy semi-topogenous order τ, a space (X, S, R) is increasing (decreasing, resp.) if and only if (X, S^c, R) is decreasing (increasing, resp.).

For a set X we denote by $O(X)$ the set of all fuzzy semi-topogenous orders on X and by $PTO(X)$ the power set of the set of all saturated fuzzy topogenous orders on X. We now recall the following operations introduced in [C2]:

(1)$^{[\,]}: O(X)\to O(X)$ is the map given by $[\tau](\alpha, \beta)=\wedge_{x, y\in X}\tau(x_{\alpha(x)}, y^{\beta(y)})$.

(2)$^{p}: O(X)\to O(X)$ is the map given by $\tau^{p}(\alpha, \beta)=\wedge_{\gamma<\alpha}\tau(\gamma, \beta)$.

(3)$^{b}: O(X)\to O(X)$ is the map given by $\tau^{b}(\alpha, \beta)=\wedge_{\gamma<\alpha, \beta<\delta}\tau(\gamma, \delta)$.

(4)$^{t}: PTO(X)\to PTO(X)$ is the map given by $A^{t}=[\vee_{\tau\in A}\tau]$.

Using these notions we have the following:

Proposition 3.5. *If a space(X, S, R) is increasing (decreasing, resp.) then (X, S^{o}, R) is also increasing (decreasing, resp.), where $^{o=[\,], p, b, t}$.*

Remark. For a saturated space (X, S, R) let $^{T, \delta, U}$ denote the fuzzy neighborhood structure, the Artico-Moresco fuzzy quasi-proximity and the Lowen fuzzy quasi-uniformity associated with S^{tp}, S^{t}, and S^{b}, respectively. If S is increasing (decreasing, resp.) then one has the following:

(1)$(x, y)\in R$ implies $\bar{\mu}^{T}(y)\leq\bar{\mu}^{T}(x)$ $(\bar{\mu}^{T}(x)\leq\bar{\mu}^{T}(y)$, *resp.*), where $\bar{\mu}^{T}(x)=1-\vee_{\tau\in S}\tau(x, 1-\mu)$.

(2)$(x, y)\in R$ implies $\bar{y}^{T}(x)=1$ $(\bar{x}^{T}(y)=1$, *resp.*).

(3)$(x, y)\in R$ implies $\delta(\alpha, y_{a})\leq\delta(\alpha, x_{a})$, $\delta(x_{a}, \alpha)\leq\delta(y_{a}, \alpha)$, $(\delta(x_{a}, \alpha)\leq\delta(y_{a}, \alpha)$, $\delta(\alpha, y_{a})\leq\delta(\alpha, x_{a})$, resp.), where $\delta(\alpha, \beta)=1-S^{t}(\alpha, 1-\beta)$ $(\alpha, \beta\in I^{X})$.

(4)$(x, y)\in R$ implies $u_{\tau}(y, z)\leq u_{\tau}(x, z)$, $u_{\tau}(z, x)\leq u_{\tau}(z, y)$, $(u_{\tau}(z, y)\leq u_{\tau}(z, x)$, $u_{\tau}(x, z)\leq u_{\tau}(y, z)$, resp.), where $u_{\tau}(x, y)=1-\tau(x, y^{0})$ for $\tau\in S$.

For any space (X, S, R),

$U(S)=\vee\{S':(X, S', R)$ is increasing and $S'\leq S\}$

is the finest of all increasing structures coarser than S and

$L(S)=\vee\{S':(X, S', R)$ is decreasing and $S'\leq S\}$

is the finest of all decreasing structures coarser than S.

Proposition 3.6. *For a space (X, S, R) one has the following*:

(1)$U(S)^{o}\leq U(S^{o})$ and $L(S)^{o}\leq L(S^{o})$ where $^{o=[\,], p, b}$.

(2)If $S\sim S^{o}$ then $U(S)\sim U(S)^{o}$ and $L(S)\sim L(S)^{o}$ where $^{o=[\,], p, b}$.

(3)$U(S)^{t}\leq U(S^{t})$ and $L(S)^{t}\leq L(S^{t})$.

(4)$L(S)^{c}\sim U(S^{c})$ and $U(S)^{c}\sim L(S^{c})$.

Proof. (1)$U(S)\leq S$ implies $U(S)^{o}\leq S^{o}$ (cf. [C2]). By Proposition 3.5 $U(S)^{o}$ is increasing and hence $U(S)^{o}\leq U(S^{o})$. The case for L is similar.

(2)It is immediate from (1) and the definition of an ordinary operation ^{o}.

(3)$U(S)\leq S$ implies $U(S)^{t}\leq S^{t}$ (cf. [C2]) and hence $U(U(S)^{t})\leq U(S^{t})$.

Since $U(S)^{t}\leq U(U(S)^{t})$ we have $U(S)^{t}\leq U(S^{t})$. The case for L is similar.

(4)Clearly $L(S)^{c}\leq U(S^{c})$ and $U(S)^{c}\leq L(S^{c})$. Let us apply these inequalities for $S'=S^{c}$. Then $U(S^{c})=U(S')=U(S')^{cc}\leq L(S'^{c})^{c}=L(S^{cc})^{c}=L(S)^{c}$ and similarly $L(S^{c})=L(S')=L(S')^{cc}\leq U(S'^{c})^{c}=U(S^{cc})^{c}=U(S)^{c}$.

Recall that for any map $f:X\to Y$ and any fuzzy semi-topogenous order τ on Y, we obtain $f^{-1}(\tau)(x_{a}, y^{b})=\tau(f(x)_{a}, f(y)^{b})$ for $x, y\in X$ and $a, b\in I$.

Using this fact the following is immediate.

Lemma 3.7. *If a map $f:(X, R)\to(Y, R')$ is an isotone and τ is increasing (decreasing, rest.) on (Y, R') then $f^{-1}(\tau)$ is increasing (decreasing, resp.) on (X, R).*

The full subcategories of [PFSyn] determined by increasing saturated spaces (decreasing saturated spaces, resp.) is denoted by [IPFSyn] ([DPFSyn], resp.).

Theorem 3.8. *The category* [IPFSyn] *is bireflective in the category* [PFSyn].

Proof. Take any $(X, S, R) \in$ [PFSyn]. Then, by Proposition 3.5 and the definition of $U(S)$, the identity map $id_X : (X, S, R) \to (X, [U(S)], R)$ is a continuous isotone and $(X, [U(S)], R) \in$ [PFSyn]. Take any increasing saturated space (Y, S', R') and any continuous isotone $f : (X, S, R) \to (Y, S', R')$. Let $g : (X, [U(S)], R) \to (Y, S', R')$ be the map f as set map. Then g is a continuous isotone. Indeed, it is clear that g is an isotone and hence $g^{-1}(S') = \{g^{-1}(\tau) : \tau \in S\}$ is increasing by Lemma 3.7. It follows from the definition of $U(S)$ that $g^{-1}(S') \leq U(S) \leq [U(S)]$ on (X, R). This means that g is continuous. Thus $id_X : (X, S, R) \to (X, [U(S)], R)$ is the [IPFSyn]-reflection of (X, S, R) in the category [PFSyn].

Remark. The functor F: [IPFSyn] $\to$ [DPFSyn] $(F(X, S, R) = (X, S^c, R), F(f) = f)$ is clearly an isomorphism. Therefore, by the above theorem [DPFSyn] is also bireflective in the category [PFSyn]. In fact $id_X : (X, S, R) \to (X, [L(S)], R)$ is the [DPFSyn]-reflection of (X, S, R) in [PFSyn].

References

[AHE] A. A. Allam, S. A. Hussein and K. El-Saady, *Fuzzy syntopogenous structures and order*, Fuzzy Sets and Systems **63** (1994), 91-98.

[AHS] J. Adamek, H. Herrlich and G. E. Strecker, *Abstract and Concrete Categories*, Wiely, New York, 1990.

[BF] D. C. J Brugess and M. Fitzpartrick, *Syntopogenous preordered spaces*, Math. Proc. Camb. Phil. Soc. **80** (1976), 71-79.

[C1] S. H. Chung, *Saturated fuzzy syntopogenous spaces*, Fuzzy sets and systems 86(1997), 107-119.

[C2] S. H. Chung, *Operations on saturated fuzzy syntopogenous structures*, J. Korean Math. Soc. 33 (1996), No4, 879-889.

[K] A. K. Katsaras, *Fuzzy syntopogenous structures compatible with Lowen fuzzy uniformities and Artico-Moresco fuzzy proximities*, Fuzzy Sets and Systems **36** (1990), 375-393.

[L1] R. Lowen, *Fuzzy neighborhood spaces*, Fuzzy Sets and Systems **7** (1982), 165-189.

[L2] R. Lowen, *Fuzzy uniform spaces*, J. Mah. Anal. Appl. **82** (1981), 370-385.

[M] K. Maltolcy, *Syntopogenous spaces with preorder I (Convexity)*, Acta Math. Hungarica **43** (1984), 347-363.

정 세 화(Sehwa Chung) 정회원
1989년 : 서강대학교 수학과 이학 박사
현재 : 경원대학교 수학과 교수
1996년 8월 : 미국 Rutgers University 수학과 방문교수

주 무 홍(Moohong Joo) 정회원
- 서울대학교 물리학과 졸업
1986년 : 단국대학교 수학과 이학 박사
현재 : 경원대학교 수학과 교수