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# Estimation of the parameters in an Exponential Distribution with Type-II Censoring

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## Abstract

In this paper, we propose the minimum risk estimator (MRE) and the approximate maximum likelihood estimator (AMLE) of the location and the scale parameters of the two-parameter exponential distribution with Type-II censoring. The MRE's can be derived by minimizing the mean squared error among the class of estimators which include some estimators as special cases. We show that the MRE's are more efficient than the other estimators of the scale and the location parameter in the terms of the mean squared error.

## 1. Introduction

The random variable $X$ has an exponential distribution if it has a probability density function (pdf) of the forms

$$f(x;\mu,\sigma)=\frac{1}{\sigma}\exp\left(-\frac{(x-\mu)}{\sigma}\right),\ 0<\ \mu<\ x,\ 0\ <\ \sigma, \tag{1.1}$$

where $\mu$ and $\sigma$ are the location and the scale parameters, respectively.

Lloyd (1952) described a method of obtaining the best linear unbiased estimators (BLUEs) of the parameters of exponential distribution, using order statistics. Gupta (1952) proposed the estimation of the mean and standard deviation of a normal population from a censored sample. Balakrishnan (1990) studied the maximum likelihood estimation of the parameters in exponential distribution based on multiplying Type-II censored sample.

The approximate maximum likelihood estimation method was first developed by Balakrishnan (1989a, b) for the purpose of providing the explicit estimators of the scale parameter in the Rayleigh distribution and the mean and standard deviation in the normal distribution with censoring. For the extreme value distribution with censoring and the half-logistic distribution with Type-II right censoring, Balakrishnan and Varadan (1990) and

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Balakrishnan and Wong (1991) provided a method of deriving explicit estimators by approximating the likelihood equation, respectively. They also studied the biases and the variances of the proposed estimators and showed that these estimators are almost as efficient as the MLEs and just as efficient as the BLUEs.

Some historical remarks and a good summary of the approximate maximum likelihood estimation may be found in Balakrishnan and Cohen (1991). Recently Kang (1996) obtained the AMLE for the scale parameter of the double exponential distribution based on Type-II censored samples and he showed that the proposed estimator is generally more efficient than the BLUE and the optimum unbiased absolute estimator.

## 2. Estimation of Two Parameters

Consider two-parameter exponential distribution with density function (1.1) and cumulative distribution function (cdf)

$$F(x) = \begin{cases} \exp\left(\frac{-(x-\mu)}{\sigma}\right), & x \geq \mu \\ 1, & x < \mu. \end{cases} \tag{2.1}$$

Let us consider an experiment in which $n$ exponential components are put to test simultaneously at time $x=0$, and the failure times of there components are recorded. Suppose some initial observations are censored (possibly because of some failures during the time when some checks and adjustments are being made on the devices) and some final observations are also censored (possibly because the experimenter terminates the experiment before all components have failed). Then let

$$X_{r+1:n} \leq X_{r+2:n} \leq \cdots \leq X_{n-s:n} \tag{2.2}$$

be the available Type-II censored sample from the exponential distribution with density function (1.1), where the first $r$ and the last $s$ observations are censored.

It is well known that the expectation, the variance, and the covariance of the $i$-th order statistic $X_{i:n}$ from the two-parameter exponential distribution with density function (1.1) are given by

$$E(X_{i:n}) = \mu + \sigma \sum_{j=1}^{i} (n-j+1)^{-1} \tag{2.3}$$

and

$$\begin{aligned} \mathrm{Var}(X_{i:n}) &= \sigma^2 \sum_{k=1}^{i} (n-k+1)^{-2} \\ &= \mathrm{Cov}(X_{i:n}, X_{j:n}), \quad i<j. \end{aligned} \tag{2.4}$$

The expectation and the variance of the $i$-th order statistic $Z_{i:n}$ from the standard exponential distribution are

$$E(Z_{i:n})=\alpha_{i:n}=\sum_{l=n-i+1}^{n}1/l \quad (1\leq i\leq n)$$

and

$$\mathrm{Cov}(Z_{i:n},Z_{j:n})=\beta_{i,j:n}=\sum_{l=n-i+1}^{n}1/l^2 \quad (1\leq i\leq j\leq n).$$

Lloyd (1952) derive the BLUEs of $\mu$ and $\sigma$ as follows;

$$\hat{\mu}_{BLUE}=\sum_{i=r+1}^{n-s}a_i X_{i:n} \tag{2.5}$$

and

$$\hat{\sigma}_{BLUE}=\sum_{i=r+1}^{n-s}b_i X_{i:n}, \tag{2.6}$$

where

$$a_i=\begin{cases}1+\dfrac{(n-r-1)}{(n-r-s-1)}\displaystyle\sum_{l=n-r}^{n}\frac{1}{l}, & \text{for } i=r+1\\ -\dfrac{1}{(n-r-s-1)}\displaystyle\sum_{l=n-r}^{n}\frac{1}{l}, & \text{for } r+2\leq i\leq n-s-1\\ -\dfrac{s+1}{n-r-s-1}\displaystyle\sum_{l=n-r}^{n}\frac{1}{l}, & \text{for } i=n-s\end{cases}$$

and

$$b_i=\begin{cases}-\dfrac{n-r-1}{n-r-s-1}, & \text{for } i=r+1\\ \dfrac{1}{n-r-s-1}, & \text{for } r+2\leq i\leq n-s-1\\ \dfrac{s+1}{n-r-s-1}, & \text{for } i=n-s.\end{cases}$$

The variances of the BLUEs (see Balakrishnan and Cohen (1991)) are given by

$$\mathrm{Var}(\hat{\mu}_{\mathrm{BLUE}})=\sigma^2\left[\frac{1}{n-r-s-1}\Big(\sum_{l=n-r}^{n}1/l\Big)^2+\sum_{l=n-r}^{n}(1/l)^2\right] \tag{2.7}$$

and

$$\mathrm{Var}(\hat{\sigma}_{\mathrm{BLUE}})=\frac{\sigma^2}{n-r-s-1}. \tag{2.8}$$

Gupta (1952) proposed the simplified linear estimators (SLE) for $\mu$ and $\sigma$ obtained from the BLUEs of $\mu$ and $\sigma$ simply by replacing the variance-covariance matrix by an identity matrix. Even though this method appears to be crude, it gives surprisingly good results in the case of the normal distribution.

The SLE of $\mu$ is given by (see Balakrishnan and Cohen (1991))

$$\hat{\mu}_{SLE} = \sum_{j=r+1}^{n-s} c_j X_{j:n}, \tag{2.9}$$

where

$$c_j = \frac{\sum_{i=r+1}^{n-s} a_{i:n}^2 - a_{j:n} \sum_{i=r+1}^{n-s} a_{i:n}}{(n-r-s) \sum_{i=r+1}^{n-s} (a_{i:n} - \bar{a})^2}$$
$$= \frac{1}{n-r-s} - \frac{\bar{a}(a_{j:n} - \bar{a})}{\sum_{i=r+1}^{n-s} (a_{i:n} - \bar{a})^2}, \qquad r+1 \le j \le n-s$$

with $\bar{a}$ as defined in

$$\bar{a} = \frac{1}{n-r-s} \sum_{i=r+1}^{n-s} a_{i:n}.$$

Similarly, the SLE of $\sigma$ is given by

$$\hat{\sigma}_{SLE} = \sum_{j=r+1}^{n-s} d_j X_{j:n}, \tag{2.10}$$

where

$$d_j = \frac{(n-r-s)a_{j:n} - \sum_{i=r+1}^{n-s} a_{i:n}}{(n-r-s) \sum_{i=r+1}^{n-s} (a_{i:n} - \bar{a})^2}$$
$$= \frac{a_{j:n} - \bar{a}}{\sum_{i=r+1}^{n-s} (a_{i:n} - \bar{a})^2}, \qquad r+1 \le j \le n-s.$$

The variances of these simplified linear estimators may be obtained as

$$\mathrm{Var}(\hat{\mu}_{SLE}) = \sigma^2 \sum_{i=r+1}^{n-s} \sum_{j=r+1}^{n-s} c_i c_j \beta_{i,j:n}. \tag{2.11}$$

$$\mathrm{Var}(\hat{\sigma}_{SLE}) = \sigma^2 \sum_{i=r+1}^{n-s} \sum_{j=r+1}^{n-s} c_i d_j \beta_{i,j:n}. \tag{2.12}$$

Now, we obtain the AMLEs of the parameters in two-parameter exponential distribution with Type-II censoring. The likelihood function based on the Type-II censored sample in (2.2) is given by

$$L = \frac{n!}{r!s!} [F(X_{r+1:n}; \mu, \sigma)]^r [1 - F(X_{n-s:n}; \mu, \sigma)]^s \prod_{i=r+1}^{n-s} f(X_{i:n}; \mu, \sigma), \quad X_{r+1:n} \ge \mu \tag{2.13}$$

which upon denoting $Z_{i:n}=(X_{i:n}-\mu)/\sigma$, can be written as

$$L=\frac{n!}{r!s!}[F(Z_{r+1:n};\mu,\sigma)]^r[1-F(Z_{n-s:n};\mu,\sigma)]^s\prod_{i=r+1}^{n-s}f(Z_{i:n};\mu,\sigma),$$
$$Z_{r+1:n}\geq 0, \tag{2.14}$$

where $A=n-r-s$ is the size of the censored sample (2.2), and $f(z)$ and $F(z)$ are the pdf and the cdf of the standard exponential distribution, respectively.

So we can obtain the AMLE of the location parameter as follows;

$$\hat{\mu}_{AMLE}=X_{r+1:n}.$$

Now, we will obtain the AMLE of the scale parameter. First, we differentiate the logarithm of the likelihood function (2.14) for $\sigma$ as follows;

$$\begin{aligned}\frac{\partial \ln L}{\partial\sigma} &= -\frac{1}{\sigma}\left[A+rZ_{r+1:n}\frac{f(Z_{r+1:n})}{F(Z_{r+1:n})}-sZ_{n-s:n}\frac{f(Z_{n-s:n})}{1-F(Z_{n-s:n})}\right.\\ &\quad\left.+\sum_{i=r+1}^{n-s}Z_{i:n}\frac{f'(Z_{i:n})}{f(Z_{i}:n)}\right]\\ &=0.\end{aligned} \tag{2.15}$$

Equation (2.15) does not admit an explict solution for $\sigma$. But $\frac{f(Z_{n-s:n})}{1-F(Z_{n-s:n})}=1$ and $\frac{f'(Z_{i:n})}{f(Z_{i:n})}=-1$, and we can expand the function $\frac{f(Z_{r+1:n})}{F(Z_{r+1:n})}$ appearing in (2.15) to Taylor series around the point $\xi_{r+1}=F^{-1}(p_{r+1})=-\ln(q_{r+1})$ and then approximate it by

$$\frac{f(Z_{r+1:n})}{F(Z_{r+1:n})}\simeq\alpha-\beta Z_{r+1:n} \tag{2.16}$$

where $p_i=\frac{i}{(n+1)}$, $q_i=1-p_i$, $\alpha=\frac{f(\xi_{r+1})}{p_{r+1}}\left[1+\xi_{r+1}+\frac{f(\xi_{r+1})}{p_{r+1}}\xi_{r+1}\right]$, and

$$\beta=\frac{f(\xi_{r+1})}{p_{r+1}^2}[p_{r+1}+f(\xi_{r+1})].$$

Now making use of the approximate expression in (2.16), we obtain the approximate likelihood equation of (2.15) as follows;

$$\begin{aligned}\frac{\partial \ln L}{\partial\sigma} &\simeq \frac{\partial \ln L^*}{\partial\sigma}\\ &= -\frac{1}{\sigma}[A+rZ_{r+1:n}(\alpha-\beta Z_{r+1:n})-sZ_{n-s:n}-\sum_{i=r+1}^{n-s}Z_{i:n}]\\ &=0.\end{aligned} \tag{2.17}$$

Upon solving equation (2.17) for $\sigma$, we derive the AMLE of $\sigma$ as follows;

$$\hat{\sigma}_{AMLE}=\frac{1}{A}\left[ sX_{n-s:n}-(n-r)X_{r+1:n}+\sum_{i=r+1}^{n-s}X_{i:n}\right]. \tag{2.18}$$

Now from equation (2.3), we can obtain the means of the AMLEs as follows;

$$E(\hat{\mu}_{AMLE})=E(X_{r+1:n})=\mu+\sigma\sum_{j=1}^{r+1}(n-j+1)^{-1} \tag{2.19}$$

and

$$\begin{aligned} E(\hat{\sigma}) &=\frac{1}{A}\left[sE(X_{n-s:n})-(n-r)E(X_{r+1:n})+\sum_{i=r+1}^{n-s}E(X_{i:n})\right] \\ &=\frac{\sigma}{A}\left[sh(n-s)-(n-r)h(r+1)+\sum_{i=r+1}^{n-s}h(i)\right], \end{aligned} \tag{2.20}$$

where $h(r)=\sum_{j=1}^{r}(n-j+1)^{-1}$. From equation (2.4), we can also obtain the variances of $\hat{\mu}_{AMLE}$ and $\hat{\sigma}_{AMLE}$ as follows;

$$\mathrm{Var}(\hat{\mu}_{AMLE})=\sigma^2\sum_{j=1}^{r+1}(n-j+1)^{-2} \tag{2.21}$$

and

$$\begin{aligned} \mathrm{Var}(\hat{\sigma}_{AMLE})= &\frac{\sigma^2}{A^2}\left[s^2g(n-s)+(2s+1)\sum_{i=r+1}^{n-s}g(i)\right. \\ &\left.+2\sum_{i=r+1}^{n-s-1}(n-s-i)g(i)-(n-r)^2g(r+1)\right], \end{aligned} \tag{2.22}$$

where $g(r)=\sum_{j=1}^{r}(n-j+1)^{-2}$.

Also we propose the minimum risk estimator (MRE) of the location parameter. The MRE can be derived by minimizing the mean squared error among the class of estimators of the form $c_1X_{r+1:n}+c_2\sum_{i=r+1}^{n-s}X_{i:n}$ where $c_1$ and $c_2$ are constants.

We can obtain the MRE of $\mu$ as follows;

$$\hat{\mu}_{MRE}=c_1X_{r+1:n}+c_2\sum_{i=r+1}^{n-s}X_{i:n}, \tag{2.23}$$

where

$$c_2=\frac{Ah(r+1)-h(r+1)\sum_{i=r+1}^{n-s}h(i)}{B},$$

$$c_1=1-Ac_2,$$

and

$$B= \sum_{i=r+1}^{n-s} g(i)+2 \sum_{i=r+1}^{n-s}(n-s-i)g(i)-2Ah(r+1) \sum_{i=r+1}^{n-s} h(i)$$
$$+\left(\sum_{i=r+1}^{n-s} h(i)\right)^2 A^2h(r+1)^2-A^2g(r+1).$$

Then the mean and the variance of the MRE of $\mu$ are given by

$$E(\hat{\mu}_{MRE})=(c_1+Ac_2)\mu+\left(c_1h(r+1)+c_2 \sum_{i=r+1}^{n-s} h(i)\right)\sigma \tag{2.24}$$

and

$$\begin{aligned}\mathrm{Var}(\hat{\mu}_{MRE})= \sigma^2\Big[& c_1^2g(r+1)+2c_1c_2Ag(r+1)+c_2^2 \sum_{i=r+1}^{n-s} g(i)\\ &+2c_2^2 \sum_{i=r+1}^{n-s}(n-s-i)g(i)\Big].\end{aligned} \tag{2.25}$$

Similarly, we propose the minimum risk estimator (MRE) of the scale parameter. The MRE can be derived by minimizing the mean squared error among the class of estimators of the form $c_{11}X_{n-s:n}+c_{22}X_{r+1:n}+c_{33} \sum_{i=r+1}^{n-s} X_{i:n}$ where $c_{11}$, $c_{22}$, and $c_{33}$ are constants.

We can obtain the MRE of $\sigma$ as follows;

$$\hat{\sigma}_{MRE}= c_{11}X_{n-s:n}+c_{22}X_{r+1:n}+c_{33} \sum_{i=r+1}^{n-s} X_{i:n}, \tag{2.23}$$

where

$$c_{33}= \frac{D(h(r+1)-h(n-s))-B\left(\sum_{i=r+1}^{n-s} h(i)-Ah(n-s)\right)}{CD-BE},$$

$$c_{22}= \frac{h(r+1)-h(n-s)-Cc_{33}}{B},$$

$$c_{11}= -c_{22}-Ac_{33},$$

$$B= g(n-s)-g(r+1)+h(r+1)^2-2h(r+1)h(n-s)+h(n-s)^2,$$

$$C= Ag(n-s)- \sum_{i=r+1}^{n-s} g(i)+h(r+1) \sum_{i=r+1}^{n-s} h(i)-Ah(r+1)h(n-s)$$
$$-h(n-s) \sum_{i=r+1}^{n-s} h(i)+Ah(n-s)^2,$$

$$D= Ag(n-s)- \sum_{i=r+1}^{n-s} g(i)+h(r+1) \sum_{i=r+1}^{n-s} h(i)-h(n-s) \sum_{i=r+1}^{n-s} h(i)$$
$$-Ah(r+1)h(n-s)+Ah(n-s)^2$$

and

$$E = A^2 g(n-s) + \sum_{i=r+1}^{n-s} g(i) + 2\sum_{i=r+1}^{n-s}(n-s-i)g(i) - 2A\sum_{i=r+1}^{n-s} h(i)$$
$$+\left(\sum_{i=r+1}^{n-s} h(i)\right)^{2-2A(n-s)} \sum_{i=r+1}^{n-s} h(i) + (Ah(n-s))^2.$$

We can obtain the mean and the variance of the MRE of $\sigma$ as follows;

$$E(\hat{\sigma}_{MRE}) = (c_{11} + c_{22} + Ac_{33})\mu + \left(c_{11}h(n-s) + c_{22}h(r+1) + c_{33}\sum_{i=r+1}^{n-s} h(i)\right)\sigma \tag{2.24}$$

and

$$\mathrm{Var}(\hat{\sigma}_{MRE}) = \sigma^2\Big[c_{11}^2 g(n-s) + c_{22}^2 g(r+1) + c_{33}^2 \sum_{i=r+1}^{n-s} g(i)$$
$$+ 2c_{33}^2 \sum_{i=r+1}^{n-s}(n-s-i)g(i) + 2c_{11}c_{22}g(r+1)$$
$$+ 2c_{22}c_{33}Ag(r+1) + 2c_{11}c_{33}\sum_{i=r+1}^{n-s} g(i)\Big]. \tag{2.25}$$

We calculate the numerical values of the variances and the relative mean squared errors of estimators for sample size $n=3(1)9$ and various choices of censoring. These values are given in Table 1.1 and 1.2.

From Table 1.1 and 1.2, we can obtain the following results; The MRE $\hat{\mu}_{MRE}$ is generally more efficient than the other estimators in the sense of MSE. The AMLE $\hat{\sigma}_{AMLE}$ is more efficient and simple than the other estimators $\hat{\sigma}_{BLUE}$ and $\hat{\sigma}_{SLE}$ in the sense of MSE. The AMLE $\hat{\sigma}_{AMLE}$ is the same as the MRE $\hat{\sigma}_{MRE}$ of the scale parameter. So the proposed AMLE $\hat{\sigma}_{AMLE}$ of the scale parameter is not only very simple estimator but also very good estimator in the sense of the MSE.

## Table 1.1.

The relative mean squared errors for the estimators of the location parameter $\mu$.

| $n$ | $r$ | $s$ | $\hat{\mu}_{BLUE}$ | $\hat{\mu}_{SLE}$ | $\hat{\mu}_{AMLE}$ | $\hat{\mu}_{MRE}$ |
|---|---|---|---|---|---|---|
| 3 | 0 | 0 | .16667 | .25850 | .22222 | .14815 |
| 4 | 0 | 0 | .08333 | .17911 | .07813 | .07813 |
| | 0 | 1 | .09375 | .11877 | .12500 | .08361 |
| | 1 | 0 | .34375 | .48724 | .51389 | .28704 |
| | 1 | 1 | .51389 | .51389 | .51389 | .34375 |
| 5 | 0 | 0 | .05000 | .13915 | .08000 | .04800 |
| | 0 | 1 | .05323 | .08491 | .08000 | .05020 |
| | 1 | 0 | .17100 | .30792 | .30500 | .15312 |
| | 1 | 1 | .20375 | .24193 | .30500 | .17091 |
| 6 | 0 | 0 | .03333 | .11483 | .05556 | .03241 |
| | 0 | 1 | .03472 | .06782 | .05556 | .03346 |
| | 0 | 2 | .03704 | .05263 | .05556 | .03499 |
| | 1 | 0 | .10139 | .22274 | .20222 | .09467 |
| | 1 | 1 | .11259 | .15697 | .20222 | .10205 |
| | 1 | 2 | .13500 | .15075 | .20222 | .11377 |
| | 2 | 0 | .25704 | .43593 | .51056 | .22535 |
| | 2 | 1 | .32042 | .39167 | .51056 | .25875 |
| | 2 | 2 | .51056 | .51056 | .51056 | .32042 |
| 7 | 0 | 0 | .02381 | .09831 | .04082 | .02332 |
| | 0 | 1 | .02449 | .05734 | .04082 | .02389 |
| | 0 | 2 | .02551 | .04339 | .04082 | .02467 |
| | 1 | 0 | .06735 | .17379 | .14399 | .06415 |
| | 1 | 1 | .07214 | .11620 | .14399 | .06778 |
| | 1 | 2 | .08012 | .10160 | .14399 | .07308 |
| | 2 | 0 | .15309 | .30598 | .34780 | .14011 |
| | 2 | 1 | .17472 | .23181 | .34780 | .15437 |
| | 2 | 2 | .21799 | .23890 | .34780 | .17700 |

**Table 1.1.**(continued)

| $n$ | $r$ | $s$ | $\hat{\mu}_{BLUE}$ | $\hat{\mu}_{SLE}$ | $\hat{\mu}_{AMLE}$ | $\hat{\mu}_{MRE}$ |
|---|---|---|---|---|---|---|
| 8 | 0 | 0 | .01786 | .08630 | .03125 | .01758 |
| | 0 | 1 | .01823 | .05015 | .03125 | .01791 |
| | 0 | 2 | .01875 | .03763 | .03125 | .01835 |
| | 1 | 0 | .04789 | .14229 | .10728 | .04628 |
| | 1 | 1 | .05038 | .09245 | .10728 | .04827 |
| | 1 | 2 | .05397 | .07743 | .10728 | .05103 |
| | 2 | 0 | .10157 | .23257 | .25262 | .09528 |
| | 2 | 1 | .11101 | .16597 | .25262 | .10242 |
| | 2 | 2 | .12675 | .15418 | .25262 | .11287 |
| 9 | 0 | 0 | .01389 | .07712 | .02469 | .01372 |
| | 0 | 1 | .01411 | .04486 | .02469 | .01392 |
| | 0 | 2 | .01440 | .03361 | .02469 | .01419 |
| | 0 | 3 | .01481 | .02731 | .02469 | .01454 |
| | 1 | 0 | .03593 | .12043 | .08372 | .03494 |
| | 1 | 1 | .03726 | .07697 | .08372 | .03612 |
| | 1 | 2 | .03912 | .06302 | .08372 | .03771 |
| | 1 | 3 | .04191 | .05662 | .08372 | .03981 |
| | 2 | 0 | .07231 | .18615 | .19200 | .06890 |
| | 2 | 1 | .07710 | .12833 | .19200 | .07287 |
| | 2 | 2 | .08428 | .11337 | .19200 | .07840 |
| | 2 | 3 | .09625 | .11221 | .19200 | .08618 |
| | 3 | 0 | .13570 | .29066 | .37387 | .12578 |
| | 3 | 1 | .15059 | .21624 | .37387 | .12578 |
| | 3 | 2 | .17540 | .20874 | .37387 | .15351 |
| | 3 | 3 | .22502 | .23821 | .37387 | .17886 |

**Table 1.2.**

The relative mean squared errors for the estimators of the scale parameter $\sigma$

| $n$ | $r$ | $s$ | $\hat{\sigma}_{BLUE}$ | $\hat{\sigma}_{SLE}$ | $\hat{\sigma}_{AMLE}$ | $\hat{\sigma}_{MRE}$ |
|---|---|---|---|---|---|---|
| 3 | 0 | 0 | .50000 | .59184 | .33333 | .33333 |
| 4 | 0 | 0 | .33333 | .42911 | .25000 | .25000 |
| | 0 | 1 | .50000 | .53463 | .33333 | .33333 |
| | 1 | 0 | .50000 | .59184 | .33333 | .33333 |
| | 1 | 1 | 1.00000 | 1.00000 | .50000 | .50000 |
| 5 | 0 | 0 | .25000 | .33915 | .20000 | .20000 |
| | 0 | 1 | .33333 | .37269 | .25000 | .25000 |
| | 1 | 0 | .33333 | .42911 | .25000 | .25000 |
| | 1 | 1 | .50000 | .53463 | .33333 | .33333 |
| 6 | 0 | 0 | .20000 | .28149 | .16667 | .16667 |
| | 0 | 1 | .25000 | .28875 | .20000 | .20000 |
| | 0 | 2 | .33333 | .35710 | .25000 | .25000 |
| | 1 | 0 | .25000 | .33915 | .20000 | .20000 |
| | 1 | 1 | .33333 | .37269 | .25000 | .25000 |
| | 1 | 2 | .50000 | .51790 | .33333 | .33333 |
| | 2 | 0 | .33333 | .42911 | .25000 | .25000 |
| | 2 | 1 | .50000 | .53463 | .33333 | .33333 |
| | 2 | 2 | 1.00000 | 1.00000 | .50000 | .50000 |
| 7 | 0 | 0 | .16667 | .24117 | .14286 | .14286 |
| | 0 | 1 | .20000 | .23690 | .16667 | .16667 |
| | 0 | 2 | .25000 | .27526 | .20000 | .20000 |
| | 1 | 0 | .20000 | .28149 | .16667 | .16667 |
| | 1 | 1 | .25000 | .28875 | .20000 | .20000 |
| | 1 | 2 | .33333 | .35710 | .25000 | .25000 |
| | 2 | 0 | .25000 | .33915 | .20000 | .20000 |
| | 2 | 1 | .33333 | .37269 | .25000 | .25000 |
| | 2 | 2 | .50000 | .51790 | .33333 | .33333 |

**Table 1.2**(continued)

| $n$ | $r$ | $s$ | $\hat{\sigma}_{BLUE}$ | $\hat{\sigma}_{SLE}$ | $\hat{\sigma}_{AMLE}$ | $\hat{\sigma}_{MRE}$ |
|---|---|---|---|---|---|---|
| 8 | 0 | 0 | .14286 | .21130 | .12500 | .12500 |
| | 0 | 1 | .16667 | .20150 | .14286 | .14286 |
| | 0 | 2 | .20000 | .22517 | .16667 | .16667 |
| | 1 | 0 | .16667 | .24117 | .14286 | .14286 |
| | 1 | 1 | .20000 | .23690 | .16667 | .16667 |
| | 1 | 2 | .25000 | .27526 | .20000 | .20000 |
| | 2 | 0 | .20000 | .28149 | .16667 | .16667 |
| | 2 | 1 | .25000 | .28875 | .20000 | .20000 |
| | 2 | 2 | .33333 | .35710 | .25000 | .25000 |
| 9 | 0 | 0 | .12500 | .18823 | .11111 | .11111 |
| | 0 | 1 | .14282 | .17570 | .12500 | .12500 |
| | 0 | 2 | .16667 | .19114 | .14286 | .14286 |
| | 0 | 3 | .20000 | .22019 | .16667 | .16667 |
| | 1 | 0 | .14286 | .21130 | .12500 | .12500 |
| | 1 | 1 | .16667 | .20150 | .14286 | .14286 |
| | 1 | 2 | .20000 | .22517 | .16667 | .16667 |
| | 1 | 3 | .25000 | .26958 | .20000 | .20000 |
| | 2 | 0 | .16667 | .24117 | .14286 | .14286 |
| | 2 | 1 | .20000 | .23690 | .16667 | .16667 |
| | 2 | 2 | .25000 | .27526 | .20000 | .20000 |
| | 2 | 3 | .33333 | .35057 | .25000 | .25000 |
| | 3 | 0 | .20000 | .28149 | .16667 | .16667 |
| | 3 | 1 | .25000 | .28875 | .20000 | .20000 |
| | 3 | 2 | .33333 | .35710 | .25000 | .25000 |
| | 3 | 3 | .50000 | .51088 | .33333 | .33333 |

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