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A Note on Almost Sure Properties of Exchangeable Random Variables*

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Abstract

In this paper a general result on almost sure properties is proved for a sequence of exchangeable random variables. Some known results follow from the theorem as special cases.

1. Introduction and Preliminaries

Random variables $X_1,\cdots,X_n$ are said to be exchangeable if the joint distribution of $(X_1,\cdots,X_n)$ is permutation invariant. A sequence of random variables $\{X_n\}$ is said to be exchangeable if every finite subset is exchangeable. For a sequence of exchangeable random variables $\{X_n\}$, Taylor and Hu [6] obtained a strong law of large numbers and Zhang and Taylor [8] obtained a law of the iterated logarithm by using de Finetti theorem. Hong and Kwon[3] showed a generalization of the Hartman-Wintner theorem and obtained the same result as Zhang and Taylor[8]. Similar results could be also obtained relating, for example, random power series and recurrence of random walk. Probably, those result could be established in other fields. However, the main result of the above paper inspire a general property when they are contrasted with the traditional results for independent, identically distributed(i.i.d) random variables. Thus, in this note, a general statement for exchangeable random variables will be developed which represents these patterns of almost sure property.

Let $\boldsymbol{F}$ be the class of one dimensional distribution function and U be the σ-field generated by the topology of weak convergence of the distribution functions. Then de Finetti's theorem [see 7] asserts that for an infinite sequence of exchangeable random variables $\{X_n\}$ there exists a probability measure μ on $(\boldsymbol{F}, U)$ such that

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$$P(g(X_1,\cdots,X_n)\in B)=\int_{\boldsymbol{F}} P_F(g(X_1,\cdots,X_n)\in B)d\mu(F)$$

for any Borel set B and any Borel function $g: \boldsymbol{R}^n \to \boldsymbol{R}$, $n \geq 1$. Moreover $P_F(g(X_1,\cdots,X_n)\in B)$ is computed under the assumption that the random variables $\{X_n\}$ are independent identically distribution (i.i.d.) with common distribution F.

2. Almost sure properties

In this section a generalized theorem on almost sure (a.s.) properties for an infinite sequence of exchangeable random variables is proved. Thus many results in the literature can be induced from the theorem.

Theorem 2.1. Let $\{X_n\}$ be arbitrary i.i.d. random variables on a probability space $(\Omega, \boldsymbol{A}, P)$ and suppose that if $\{X_n\}$ satisfy (H), then property (L) holds F-a.s. Let $\{Z_n\}$ be exchangeable random variables and assume that $\{Z_n\}$ satisfy (H) μ-a.s. Then, $\{Z_n\}$ satisfy property (L) F-a.s.

Proof. For any $F\in \boldsymbol{F}$, by de Finetti's theorem and assumption,

$$P_F\{\{Z_n\} \text{ satisfy } (H)\} \leq P_F\{\{Z_n\} \text{ hold } (L)\}$$

and

$$P_F\{\{Z_n\} \text{ satisfy } (H)\} = 1 \quad \mu-\text{ a.s.}$$

Thus,

$$\begin{aligned} P\{\{Z_n\} \text{ hold } L\} &= \int_{\boldsymbol{F}} P_F\{\{Z_n\} \text{ hold } (L)\}\, d\mu(F) \\ &\geq \int_{\boldsymbol{F}} P_F\{\{Z_n\} \text{ satisfy } (H)\}\, d\mu(F) \\ &= 1, \end{aligned}$$

and hence the proof is complete.

Example 2.1. Let $f(z,\omega) = \sum_{n=0}^{\infty} a_n(\omega)Z^n$ be a random power series and suppose that the absolute variables $|a_n|$ of the coefficient are exchangeable random variables. Then, by Theorem 5.4.1[4, p127] and Theorem 2.1,

$$P(r(w) = 1) = 1 \text{ if } \int_1^{\infty} \log x \, dF(x) < \infty \quad \mu-\text{a.s.}$$

and

$$P(r(w) = 0) = 1 \text{ if } \int_1^{\infty} \log x \, dF(x) = \infty \quad \mu-\text{a.s.}$$

where $r(\omega)=(\limsup |a_n(\omega)|^{\frac{1}{n}})^{-1}$ is the radius of convergence of the random power series $f(z,\omega)$.

Example 2.2. Let $\{X_n\}$ be a sequence of exchangeable random variables and let $S_n = \sum_{i=1}^{n} X_i$. Then, $p\{|S_n| < \varepsilon\ i.o.\} = 1$ for every $\varepsilon > 0$ if and only if

$$\limsup_{\rho \uparrow 1} \int_{-\alpha}^{\alpha} \frac{1}{1 - \rho\phi_{F(u)}} du = \infty \quad \mu-\text{a.s.} \qquad \text{for some } \alpha > 0,$$

where $\phi_F(u) = \int e^{ux} dF(x)$.

Proof. The "if" implication is immediate from the result in [1] and Theorem 2.1.

Now let $\int_F P_F\{|S_n| < \varepsilon \text{ i.o.}\} d\mu(F) = P\{|S_n| < \varepsilon \text{ i.o.}\} = 1$ for every $\varepsilon > 0$. Then $P_F\{|S_n| < \varepsilon \text{ i.o.}\} = 1\ \ \mu-\text{a.s.}$ for every $\varepsilon > 0$, which imply $\lim\text{sum}_{\rho \uparrow 1} \int_{-\alpha}^{\alpha} \frac{1}{1 - \rho\phi_F(u)} du = \infty$ for some $\alpha > 0$ $\mu-\text{a.s.}$ by the result in [1] for i.i.d. random variables.

Example 2.3. (c.f. Taylor and Hu [6]) Let $\{X_n\}$ be a sequence of exchangeable random variables such that $E_F|X_1| < \infty\ \ \mu-\text{a.s.}$ Then

$$E_F X_1 = \mu - \text{ a.s. if and only if } \sum_{j=1}^{n} X_j / n \to 0 \text{ a.s.}$$

Proof. The "only if" implication is immediate from Theorem 2.1. Suppose that

$\int_F P_F\{\sum_{j=1}^{n} X_j / n \to 0\} d\mu(F) = P\{\sum_{j=1}^{n} X_j / n \to 0\} = 1$. Then $P_F\{\sum_{j=1}^{n} X_j / n \to 0\} = 1\ \ \mu-\text{a.s.}$, and hence by the classical SLLN for i.i.d. random variables $E_F X_1 = 0\ \ \mu-\text{a.s.}$ This proves the example.

Example 2.4. (c.f. Hong and Kwon [3] and Zhang and Taylor [8]) Let $\{X_n\}$ be a sequence of exchangeable random variables. Then for a constant $0 < \sigma < \infty$,

$$\limsup_{n \to \infty} \sum_{j=1}^{n} X_i / \sqrt{2n \log\log n} = \sigma \quad \text{a.s.}$$

if and only if $E_F X_1 = 0$ and $E_F(X_1 - E_F X_1)^2 = \sigma^2\ \ \mu-\text{a.s.}$

Proof. The "if" implication is immediate. We now assume that

$$\int_F P_F\{\limsup_{n\to\infty} \sum_{j=1}^n X_j / \sqrt{2n\log\log n} = \sigma\} d\mu(F)$$
$$= P\{\limsup_{n\to\infty} \sum_{j=1}^n X_j / \sqrt{2n\log\log n} = \sigma\}$$
$$=1.$$

Then $P_F\{\limsup_{n\to\infty} \sum_{j=1}^n X_j / \sqrt{2n\log\log n} = \sigma\} = 1$ μ −a.s. and this implies $E_F X_1 = 0$ and $E_F(X_1)^2 = \sigma^2$ μ −a.s. by the result in [5].

3. Concluding remark

In this paper, we proved a short theorem concerning exchangeable random variables and, using the theorem and classical results for i.i.d. random variables, we give four examples which are generalizations of some classical results for i.i.d. random variables. However many classical results for i.i.d. random variables will be also extended to exchangeable random variables by the same idea in this paper.

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