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The Construction of an Efficient Incomplete Block Design by Almost Orthogonal Latin Squares of Order 6

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Abstract

The lattice designs have proved efficient but they are not always available. This article proposes an alternative, an almost lattice design, of the triple lattice design ($v=36$, $k=6$, $r=4$) which is not available. Here, we compare the almost lattice design to the a-design ($v=36$, $k=6$, $r=4$) which is another alternative of the triple lattice design ($v=36$, $k=6$, $r=4$). Consequently, we show the almost lattice design is a more efficient alternative than the a-design through A-, D-, and E-optimality.

1. Introduction

It is well-known in the statistical literature that for p equal to a prime number or a power of a prime, it is possible to construct $p-1$ mutually orthogonal latin squares. It is also well-known that one can construct efficient incomplete block designs, known as lattice designs for p^2 treatments in blocks of size p by working from orthogonal latin squares. For example, one can obtain a triple lattice design, 25 treatments in 15 blocks of size 5, by working from a 5×5 latin square, a quadruple lattice in 20 blocks by working from two mutually orthogonal 5×5 latin squares, and so on. For the case with 36 treatments this sort of thing is not possible beyond the triple lattice with 18 blocks, since it is not possible to find even one pair of mutually orthogonal 6×6 latin squares.

The object of this note is to illustrate that for 36 treatments in 24 blocks of size 6, there are two reasonable alternatives. First of all, we can work from the a-array ($k=6$, $r=4$),

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a − array

0	0	0	0
0	1	5	4
0	3	2	5
0	2	3	1
0	4	1	2
0	5	1	3

and construct an a-design with the parameters $v=36$, $k=6$, and $r=4$, where v is the number of treatments, k is the number of observations per block, and r is the number of replications. The main advantage of a-designs is that these are resolvable and also available for all situations with $v=ks$, where s is the number of blocks per replication.

It is already known that the poorest a-design is the case with $s=k=6$. The best alternative is a 6×6 triple lattice based on a 6×6 latin square. Note that in place of the $s=k=5$ or 7 and $r=3$ plans, it is recommended that one should use the lattice designs based on mutually orthogonal latin squares which are known to be efficient.

As an alternative of the a-design ($v=36$, $k=6$, $r=4$), we can begin with the following two 6×6 latin squares taken from Horton (1974)[4] and Street and Street (1987)[9], which come very close to being orthogonal.

Square I

5	5	2	3	0	1
1	0	5	4	2	3
5	4	0	1	3	2
3	2	4	5	1	0
0	3	1	2	4	5
2	1	3	0	5	4

Square II

0	1	4	5	2	3
5	4	0	1	3	2
3	2	5	4	0	1
4	5	3	2	1	0
1	3	2	0	4	5
2	0	1	3	5	4

Treat them as though they were orthogonal and proceed to set up an incomplete block design for 36 treatments in 24 blocks of size 6. Let us define this second plan as an almost lattice design (*ALD*).

As a result we have the following plans I and II which were constructed from the a-array ($k=6$, $r=4$) and the pair of almost orthogonal latin squares, respectively. We will examine the efficiencies of these two designs.

Plan I

block 1	0	6	12	18	24	30
block 2	1	7	13	19	25	31
block 3	2	8	14	20	26	32
block 4	3	9	15	21	27	33
block 5	4	10	16	22	28	34
block 6	5	11	17	23	29	35
block 1	0	7	15	20	28	35
block 2	1	8	16	21	29	30
block 3	2	9	17	22	24	31
block 4	3	10	12	23	25	32
block 5	4	11	13	18	26	33
block 6	5	6	14	19	27	34
block 1	0	11	14	21	25	31
block 2	1	6	15	22	26	32
block 3	2	7	16	23	27	33
block 4	3	8	17	18	28	34
block 5	4	9	12	19	29	35
block 6	5	10	13	20	24	30
block 1	0	10	17	19	26	33
block 2	1	11	12	20	27	34
block 3	2	6	13	21	28	35
block 4	3	7	14	22	29	30
block 5	4	8	15	23	24	31
block 6	5	9	16	18	25	32

Plan II

block 1	0	1	2	3	4	5
block 2	6	7	8	9	10	11
block 3	12	13	14	15	16	17
block 4	18	19	20	21	22	23
block 5	24	25	26	27	28	29
block 6	30	31	32	33	34	35
block 1	0	6	12	18	24	30
block 2	1	7	13	19	25	31
block 3	2	8	14	20	26	32
block 4	3	9	15	21	27	33
block 5	4	10	16	22	28	34
block 6	5	11	17	23	29	35
block 1	24	7	14	33	4	23
block 2	6	31	26	15	22	5
block 3	30	19	2	27	10	17
block 4	18	25	32	3	16	11
block 5	0	13	20	9	28	35
block 6	12	1	8	21	34	29
block 1	0	31	8	27	16	23
block 2	24	1	32	9	22	17
block 3	30	13	26	21	4	11
block 4	12	25	20	33	10	5
block 5	18	7	2	15	28	35
block 6	6	19	14	3	34	29

2. Optimality Criteria

For convenience, let us consider the common linear model $\boldsymbol{y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}$, where $\boldsymbol{y}$ represents an $n \times 1$ observation vector, $\boldsymbol{X}$ represents an $n \times p$ matrix of given elements, the vector $\boldsymbol{\beta}$ represents a $p \times 1$ vector of unknown parameters to be estimated, and the term $\boldsymbol{\varepsilon}$ is an $n \times 1$ vector of independently and identically distributed random variables with a mean of zero and an unknown variance of σ^2.

In order to compare these two designs, we consider the variance-covariance matrix of the set of parameters as a possible comparison measure. In fact, a confidence region for a set of parameters can be derived from the variance-covariance matrix. We prefer the smaller volume of the confidence region of parameters, since we need the best precision of estimation of the parameters. We may use the well-known classical criteria to compare these two plans: the determinant criterion, the smallest/largest-eigenvalue criterion, and the trace criterion. All three optimality criteria have a good geometric interpretation and statistical meaning[5][6].

The determinant is proportional to the volume of the ellipsoid under normality. The minimization of the confidence region can be obtained by minimizing the determinant of the $(X'X)^{-1}$ or by maximizing the determinant of the $X'X$. Under multivariate normality, D-optimal designs minimize the volume of the confidence ellipsoid. Because of the good statistical interpretation of D-optimality criterion, many researchers prefer this criterion, but there are some disadvantages. For example, the volume of the confidence ellipsoid may be small while there is a linear function which is estimated with a very large variance.

The trace is proportional to the sum of squares of lengths of the principal axes. A-optimal designs, based on $trace(X'X)^{-1}$, minimize the sum of the variances of the parameter estimates, but do not consider the correlations between the estimates.

The maximum eigenvalue is proportional to the square of the maximum diameter. E-optimal designs minimize max(*eigenvalues* of $X'X$). This is the equivalent to minimizing the largest variance of the estimated parameters. As we know, there is no dominant optimality. Consequently we consider various optimality criteria to compare the designs.

3. Optimality Computations

3.1 Computational Method 1

The model can be rewritten in matrix notation as $y = 1\mu + X_1\beta_1 + X_2\beta_2 + \varepsilon$ where 1 is a column of ones and where X_1 and X_2 are $n \times v$ and $n \times b$ design matrices for treatments and blocks, respectively.

From the model, the least square estimator of the parameter $\beta = (\mu|\beta_1'|\beta_2')'$ is obtained by minimizing the error sum of squares $\varepsilon'\varepsilon$ with respect to β. We can derive the normal equations $X'X\beta = X'y$ which do not have an unique solution because they are less than full rank. In our case, we can simplify the normal equations since only the treatment parameters are of our interest. The reduced normal equations for the treatment parameters can be achieved by eliminating μ and β_2, mean and block parameters, respectively from the full set of normal equations.

The reduced normal equation is $Q\beta_1 = q$,

where $Q = X_1'X_1 - X_1'X_2(X_2'X_2)^{-1}X_2'X_1$

and

$q = X_1'y - X_1'X_2(X_2'X_2)^{-1}X'_2y$.

The $v \times v$ matrix Q and the vector q are called the information matrix of the design and the vector of adjusted treatment totals, respectively. Let $a_1, \ldots, a_v$ be the eigenvalues of Q

whose eigenvectors are $p_1, \ldots, p_v$
where

$$p_i' p_j = \begin{cases} 1 & \text{if } i = j \\ 0 & otherwise \end{cases}$$

then, we can define the matrix Q as canonical form,

$$Q = \sum_{i=1}^{v} a_i p_i p_i' .$$

Since Q is a singular matrix of rank v, setting $a_v = 0$ without loss of generality, a generalized inverse of Q can be given by

$$Q^- = \sum_{i=1}^{v-1} a_i^{-1} p_i p_i'$$

Now we can define the canonical efficiency factors as $CEF_i = a_i / r$ where r is the number of replications. These canonical efficiency factors are an useful tool for selecting a design.

The three best known optimality criteria are,
(i) A-optimality: a design which has the largest harmonic mean of the *CEF*s
(ii) D-optimality: a design which has the largest geometric mean(or product) of the *CEF*s
(iii) E-optimality: a design whose smallest *CEF* is the largest *CEF* among those of any other designs.

The following Table 1 showes us the computational results from the method 1.

Table 1: Optimality based on the Computational Method 1

	Plan I	Plan II
A-optimality	0.835987	0.839329
D-optimality	0.002943	0.003131
E-optimality	0.666666	0.666666

3.2 Computational Method 2

Without a loss of generality, let us assume that the information matrix $X'X$ is nonsingular. Let $e_1, \ldots, e_p$ be the eigenvalues of $X'X$. Again $1/e_i$, $i = 1, \ldots, p$, are the eigenvalues of $(X'X)^{-1}$. We may define three classical criteria as:

(i) A-optimality= $trace(X'X)^{-1} = \sum(1/e_i)$.

(ii) D-optimality= $det(X'X)^{-1} = \prod(1/e_i)$.

(iii) E-optimality=max($1/e_1, \ldots, 1/e_p$).

Also, we have to notice that there is no unified definition of each optimality. The definition of optimality can be modified for better statistical and geometrical interpretations. For example SAS PROC OPTEX uses the following A- and D-optimality definition:

(i) A-optimality $= 100 \times [\,(p/N)/trace(X'X)^{-1}]$

(ii) D-optimality $= 100 \times (|\,X'X\,|^{1/p}/N)$

where p is the number of parameters in the linear model and N is the number of design points. For another example, D-optimality can be defined as $-\log det(X'X)$.

To compare our two designs, we create the full rank design matrices. In order to get the full rank design matrices, we took the last column off and subtract it from the other columns of the result. In fact, SAS OPTEX procedure provides three different design codings. But it is a well-known fact that A- and E-optimality are not invariant to design matrices. Different designs will be optimal with different design matrices. The D-optimality is invariant to nonsingular transformation of design matrices. The following three tables contain the results of the computations of optimality criteria from the method 2.

Table 1. D-optimality based on the Computational Method 2

D-optimality	Plan I	Plan II
$det(X'X)^{-1}$	2.929E-42	2.752E-42
$det(X'X)$	3.4144E41	3.6332E41
$-\log det(X'X)$	95.63398	95.69609
$100 \times (|\,X'X\,|^{1/p}/N)$	3.512303	3.516002

Table 2. A-optimality based on the Computational Method 2

A-optimality	Plan I	Plan II
$trace(X'X)^{-1}$	14.951157	14.883449
$100 \times [\,(p/N)/trace(X'X)^{-1}]$	2.7404047	2.7528715

Table 3. E-optimality based on the Computational Method 2

E-optimality	Plan I	Plan II
$\max(1/e_1, \ldots, 1/e_p)$	0.5	0.4678236

4. Summary and Conclusions

From Table 1 of the method 1, we can conclude the almost lattice design ($v=36$, $k=6$, $r=4$) is better than the α-design ($v=36$, $k=6$, $r=4$) when we use A- and D-optimality as decision criteria. But E-optimality doesn't tell which design we have to choose.

From the method 2, the almost lattice design ($v=36$, $k=6$, $r=4$) is more efficient than the α-design ($v=36$, $k=6$, $r=4$) based on the first D-optimality of Table 1, since we are trying to find a smaller value of the determinant of $(\boldsymbol{X}'\boldsymbol{X})^{-1}$. For $det(\boldsymbol{X}'\boldsymbol{X})$, we are looking for a larger value which is the optimality of the almost lattice design. But if we apply the last D-optimality criterion of Table 1, we reach a different conclusion. Since $|\boldsymbol{X}'\boldsymbol{X}|^{1/p}$ is the same as the geometric mean of the eigenvalues of $\boldsymbol{X}'\boldsymbol{X}$, we have to choose a larger value which is for the almost lattice design. In addition, when we use a third D-optimality criterion, we prefer the almost lattice design, since we want to have a larger value. Finally, the almost lattice design is more efficient when we use D-optimality criterion. From Table 2, the α-design ($v=36$, $k=6$, $r=4$) is worse than the almost lattice design, since we need a small value. In these A-optimalities, we prefer the almost lattice design. Finally, Table 3 tells us that the almost lattice design is better than the α-design, since we prefer the design with a smaller value.

These three tables suggest the almost lattice design unanimously. In fact, when we closely look at these three tables, we can not find any great difference between the two designs based on A-, D-, and E-optimality. We may use any one of these two designs in a real experimental situation, since there is no great efficiency difference. But, as we know, the construction of the almost lattice design is easier than the α-design. Researchers may prefer the almost lattice design to the α-design.

In conclusion, in place of the α-design ($v=36$, $k=6$, $r=4$) we recommend the use of the almost lattice design ($v=36$, $k=6$, $r=4$) as an alternative to the lattice-design ($v=36$, $k=6$, $r=4$) which is not available in place of the α-design ($v=36$, $k=6$, $r=4$).

References

[1] Atkinson, A. C. and Donev, A. N. (1992). *Optimum Experimental Designs*, New York: Oxford University Press.

[2] Cook, R. D. and Nachtsheim, C. J. (1980). "A Comparison of Algorithms for Constructing Exact D-Optiaml Designs", *Technometric*, Vol. 22, 315 - 324.

[3] Giesbrecht F. G. (1995). Lecture Note on Experimental Designs, Unpublished, Department of Statistics, North Carolina State University.

[4] Horton J. D. (1974). "Sub-Latin squares and incomplete orthogonal arrays," *Journal of Combinatorial Theory*, A 16, 23-33

[5] Mitchell T. J. (1974). "An Algorithm for the Construction of D-Optimal Experimental Designs," *Technometric*, Vol. 20. 203-210

[6] Pazman, Andrej (1986). *Foundations of Optimum Experimental Design*, D. Reidel Publishing Company

[7] Pukelsheim, Friedrich (1993). *Optimum Design of Experiments*, New York: John Wiley and Sons, Inc.

[8] SAS Institute Inc., SAS/QC Software, Version 6.10 "The Optex Procedure Draft Manual".

[9] Street A. P. and Street, D. J. (1987). *Combinatorics of Experimental Design*, Oxford Science Publications