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Some New Methods for Constructing Nested Balanced Ternary Designs[1]

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Abstract

Some new methods of constructing balanced ternary design which satisfies nestedness are presented. And its applications are also discussed and illustrated.

1. Introduction

Nested balanced incomplete block designs(NBIBD) were introduced by Preece(1967) for situation where there are two sources of variation and one source is nested within the other.

In the case of two systems of blocks, the second is nested within the first such that ignoring one of them (super-blocks or sub-blocks) leaves another as balanced incomplete block designs(BIBD).

Balanced n-ary designs were introduced by Tocher(1952) as a generalization of BIB designs. A balanced n-ary design is an arrangement V treatments in B blocks, each of size K such that i) every treatment occurs at most n-1 times in a block, ii) every treatment is replicated R times (in case of equireplicate), iii) the off-diagonal term of concurrence matrix (NN')is constant, say Λ. When n=3, we use term "balanced ternary design".

Let N be an incidence matrix of a balanced ternary (generally n-ary) design D with parameter V^*, B^*, R^*, K^*, Λ^* for V^* treatment in B^* blocks, each of size K^*, each treatment being replicated R^* times, where Λ^* denotes the constant off-diagonal element of NN'.

For a balanced ternary design(BTD) D, the design D is called a nested balanced ternary design(nested BTD) if super block of D can be split-up in ,say, s sub-blocks such that the sub-blocks themselves constitute variance balanced designs(where C-marix is completely symmetric, i.e., $a_1 I_v + a_2 1_v 1'_v$ for some a_1 and a_2). Thus, the sub-design may be a variance

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balanced designs with unequal block size. Let N_i be the incidence matrix corresponding to the set of blocks with block size k_i, $N_i'1 = k_i 1_b$, $i = 1, 2, \ldots s$, then $N = \sum_{i=1}^{s} N_i$.

Methods of constructing balanced n-ary designs have been given by Agarwal and Kumar(1986), Dey(1970), Dey, Das and Banerjee(1986), Donavan(1988), Kagegama(1979), Khodkar(1992), Morgan(1997), Sarvate and Sererry(1993), Shah and Gujarathi(1989), Shah & Dey(1972), among others.

2. Some Background for Construction

2.1 NBIBD

NBIBD has two system of block, the second system is nested within the first system (each block from the first system, called super-blocks, containing some blocks, called sub-blocks, from second system) such that ignoring either system leaves a BIBD whose blocks are those of the other system.

Example 1. Preece(1967)

Consider the initial block (2 5),(3 4) for constructing a BIBD with parameters $r=4$, $v=5$, $k_1=4$, $k_2=2$. We have the following result by cyclic developme($d=2$, mod 5).

{ (2 5) (3 4) },
{ (4 2) (5 1) }
{ (1 4) (2 3) },
{ (3 1) (4 5) },
{ (5 3) (1 2) }.

Ignoring the sub-block leaves a BIBD with parameters $v=5$, $r=4$, $b_1=5$, $k_1=4$, $\lambda_1=3$. Ignoring the super-block leaves a BIBD with parameters $v=5$, $r=4$, $b_2=10$, $k_2=2$, $\lambda_2=1$. We can get a NBIBD($v=5, r=4, b_1=5, k_1=4, \lambda_1=3, b_2=10, k_2=2, \lambda_2=1, m=2$).

2.2 Some Methods for Constructing of BTD

Consider an affine α-resolvable BIBD with parameter v^*, b^*, r^*, k^*, and λ^*. In the case of affine α-resolvable designs, we know the number of common treatments in any pair of blocks of the same group is same (say q_1), otherwise the number of common treatments in any pair of blocks of the different group is also same(say q_2).

Next, construct group-divisible design(GDD) with parameter $v' = m' \times n', b', k', r', \lambda_1, \lambda_2$ from affine α-resolvable BIBD by collapsing all possible block two by two, taking one from one group and the other from another group. Here, if the block numbers of the affine α

-resolvable BIBD are identified as treatments, this design constructed by this block numbers is a GDD.

If N_1 be the incidence matrix of an affine α-resolvable BIBD and N_2 be the incidence matrix of a GDD (we notice that the treatment number of GDD equal to block number of affine α-resolvable BIBD), then $N=N_1 \times N_2$ is incidence matrix of a BTD. Dey(1970) proposed this method of constructing and proved balancedness of this design.

Example 2. Dey(1970)

Consider the following affine α-resolvable BIBD with parameters $v^*=4$, $b^*=6$, $r^*=3$, $k^*=2$, $\lambda^*=1$.

Block no	Contents	Group
1	1 2	1
2	3 4	
3	1 3	2
4	2 4	
5	1 4	3
6	2 3	

However, this affine α-resolvable BIBD can be constructed by using a complete set of mutually orthogonal latin square (MOLS) ;

$$\begin{pmatrix} A & B \\ B & A \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix},$$

where group 1 is constructed by row as block, group 2 is constructed by column as block, and group 3 is constructed by labels as block(A's → block 5, B's → block 6).

The incidence matrix of this affine α-resolvable BIBD is as follows :

$$N_1=\begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{pmatrix}.$$

We now construct a GDD from above design. Let us collapse the blocks from the affine α-resolvable BIBD by the explained methods (two by two, taking one from one group and other from another group), The result is as follows :

$$(1\ 3)\ (1\ 4)\ (1\ 5)\ (1\ 6)\ (2\ 3)\ (2\ 4)$$
$$(2\ 5)\ (2\ 6)\ (3\ 5)\ (3\ 6)\ (4\ 5)\ (4\ 6).$$

As mentioned earlier, two block numbers are dealt with treatments. This design is GDD with parameters $v'=b=6$, $b'=n^2(t-1)t/2=12$, $r'=n(t-1)=4$, $k'=2$ $\lambda_1=0$, $\lambda_2=1$,

$m' = t = 3$, $n' = b/t = 2$, where t is number of group and n is number of blocks within group.

And the incidence matrix of this GDD is as follows :

$$N_2 = \begin{pmatrix} 1&1&1&1&0&0&0&0&0&0&0&0 \\ 0&0&0&0&1&1&1&1&0&0&0&0 \\ 1&0&0&0&1&0&0&0&1&1&0&0 \\ 0&1&0&0&0&1&0&0&0&0&1&1 \\ 0&0&1&0&0&0&1&0&1&0&1&0 \\ 0&0&0&1&0&0&0&1&0&1&0&1 \end{pmatrix}.$$

Because the condition ($v' = b$) is satisfied, $N = N_1 \times N_2$ is the incidence matrix of a BTD :

that is,
$$N = \begin{pmatrix} 2&1&2&1&1&0&1&0&2&1&1&0 \\ 1&2&1&2&0&1&0&1&0&1&1&2 \\ 1&0&0&0&2&1&1&2&1&2&0&1 \\ 0&1&1&1&1&2&2&1&1&0&2&1 \end{pmatrix}.$$

Allocation and the C-matrix are as follows :

1	1	1	1	1	2	1	2	1	1	1	2
1	2	1	2	3	3	3	3	1	2	2	2
2	2	2	2	3	4	4	3	3	3	4	3
3	4	4	4	4	4	4	4	4	3	4	4

(columns are blocks)

$$C = rI - \frac{1}{k} NN' = 10\left(I - \frac{1}{4} J\right).$$

The off-diagonal terms are all constant and every treatment occurs 2 times in a block. So this is a BTD.

Shah(1989) proposed another method of constructing BTD by using a complete set of MOLS. Consider a complete set of MOLS of order v ($v = s^m$) where s is a prime power which is constructed by Raghavarao(1971) methods, constructing by keeping zeroth row constant and taking cyclic permutations of all remaining rows. We know that $(v-1)$MOLS exists, place these $(v-1)$MOLS in a row so as to get an array of order $v \times v(v-1)$. We can get a balanced n-ary design from this array. Shah(1989) proved that if N^* is a matrix of order $v^* \times b^*$ obtained by replacing v_α symbols of $v \times v(v-1)$ array of v symbols by α, $\alpha = 0, 1, 2, \ldots, n-1$, such that $v = \sum_{\alpha=0}^{n-1} v_\alpha$, then N^* is the incidence matrix of a balanced n-ary design with parameters $V = v$, $B = v\ (v-1)$, $R = (v-1) \sum_{\alpha=0}^{n-1} \alpha v_\alpha$

$K=\sum_{\alpha=0}^{n-1} a v_{\alpha}$, $\Lambda=\left(\sum_{\alpha=0}^{n-1} a v_{\alpha}\right)^2-\sum_{\alpha=0}^{n-1} a^2 v_{\alpha}$. If we choice $\alpha=2$, we can get a BTD.

Example 3. Let $v=4$. A complete set of MOLS is as follows :

A	B	C	D		A	B	C	D		A	B	C	D
B	C	D	A		C	D	A	B		D	A	B	C
C	D	A	B		D	A	B	C		B	C	D	A
D	A	B	C		B	C	D	A		C	D	A	B .

If we replace $v_0=1$ symbol (A) by 0, $v_1=2$ symbol (B, C) by 1, $v_2=1$ symbol (D) by 2 such that $\sum_{\alpha=0}^{2} v_{\alpha}=4=v$, then incidence matrix N^* is constructed as follows :

$$N^*=\begin{pmatrix} 0 & 1 & 1 & 2 & 0 & 1 & 1 & 2 & 0 & 1 & 1 & 2 \\ 1 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 1 \\ 1 & 2 & 0 & 1 & 2 & 0 & 1 & 1 & 1 & 1 & 2 & 0 \\ 2 & 0 & 1 & 1 & 1 & 1 & 2 & 0 & 1 & 2 & 0 & 1 \end{pmatrix}.$$

Allocation and the C-matrix are as follows :

2	1	1	1	2	1	1	1	2	1	1	1
3	2	2	1	3	2	3	1	2	3	2	1
4	3	2	3	3	2	4	2	3	4	3	2
4	2	4	4	4	4	4	3	4	4	3	4

(columns are blocks)

$$C=105\left(\frac{1}{10}I-\frac{1}{42}J\right).$$

We know that the design of the incidence matrix N^* is a BTD having parameters, $V^*=4$, $B^*=v(v-1)=12$, $R^*=(v-1)\sum_{\alpha=0}^{2} a v_{\alpha}=12$, $K^*=\sum_{\alpha=0}^{2} a v_{\alpha}=4$ $\Lambda^*=\left(\sum_{\alpha=0}^{2} a v_{\alpha}\right)^2-\sum_{\alpha=0}^{2} a^2 v_{\alpha}=10$, where $v_0=1$, $v_1=2$, $v_2=1$. Also, we can choice v_{α} by another way (for example, $v_0=1$ symbol (A) by 0, $v_1=1$ symbol (B) by 1, $v_2=1$ symbol (C) by 2, $v_3=1$ symbol (D) by 3). We can get balanced n-ary designs($n\geq 4$).

3. Construction of Nested BTD

If there exist two different BIBD, Majindar(1978) showed that the another new BIBD can be producted by the all entries (entry 1) of the incidence matrix of one BIBD are replaced by all rows of the incidence matrix of another BIBD (in case of 0 replace all entries equal to 0 by the vector $(0, 0, \cdots, 0)$).

Example 4. Consider the following BIBD with parameters $v_1 = b_1 = 4$, $r_1 = k_1 = 3$, $\lambda_1 = 2$:

blocks	contents
1	1 2 3
2	1 2 4
3	1 3 4
4	2 3 4

Then, the incidence matrix, $N_1 = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}$.

And consider the following another BIBD with parameters $v_2 = b_2 = 3$, $r_2 = k_2 = 2$, $\lambda_2 = 1$:

blocks	contents
1	1 2
2	1 3
3	2 3

Then the incidence matrix, $N_2 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$.

A new BIBD can be constructed from the above two BIBD's. The design has parameters with $v = v_1$, $k = k_2 = 2$, $r = r_1 \times r_2 = 6$, $b = b_1 \times b_2 = 12$, $\lambda = \lambda_1 \times \lambda_2 = 2$.
And the incidence matrix,

$$N = \begin{pmatrix} 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \end{pmatrix}.$$

Donavan(1988) used similar methods for constructing BTD's. Consider BIBD $(v_1, b_1, r_1, k_1, \lambda_1)$ and BTD $(V_2^* = k_1, B_2^* : \rho_1, \rho_2, R_2^*, K_2^*, \Lambda_2^*)$. Donavan proved that another new BTD can be producted by using similar majindar algorithm. The new BTD has parameters with $V^* = v_1$, $B^* = b_1 \times B_2^*$, $\rho_1^* = r_1 \times \rho_1$, $\rho_2^* = r_2 \times \rho_2$, $R^* = r_1 \times R_2^*$, $K^* = K_2^*$, $\Lambda^* = \lambda_1 \times \Lambda_2^*$.

Example 5. Consider the following BIBD with parameters $v_1 = b_1 = 7$, $r_1 = k_1 = 4$, $\lambda_1 = 2$:

blocks	contents
1	3 4 6 7
2	1 2 6 7
3	2 4 5 7
4	1 3 5 7
5	2 3 5 6
6	1 4 5 6
7	1 2 3 4

Then, the incidence matrix, $N_1 = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \end{pmatrix}$.

And the consider following BTD with parameters $V_2^* = k_1 = 4$, $B_2^* = 12$, $\rho_1 = 6$, $\rho_2 = 3$, $K_2^* = 4$, $\Lambda_2^* = 10$:

blocks	contents
1	1 1 2 3
2	1 2 2 4
3	1 1 2 4
4	1 2 2 3
5	1 3 3 4
6	2 3 4 4
7	1 3 4 4
8	2 3 3 4
9	1 1 3 4
10	1 2 3 3
11	1 2 4 4
12	2 2 3 4

Then, the incidence matrix, $N_2=\begin{pmatrix} 2&1&2&1&1&0&1&0&2&1&1&0 \\ 1&2&1&2&0&1&0&1&0&1&1&2 \\ 1&0&0&1&2&1&1&2&1&2&0&1 \\ 0&1&1&0&1&2&2&1&1&0&2&1 \end{pmatrix}$.

Thus, we can get a new BTD with parameters $V^*=v_1=4$, $B^*=b_1\times B_2^*=84$, $\rho_1^*=r_1\times\rho_1=24$, $\rho_2^*=r_1\times\rho_2=12$, $R^*=r_1\times R_2^*=48$, $K^*=K_2^*=4$, $\Lambda^*=\lambda_1\times\Lambda_2^*=20$.
And the incidence matrix,

$$N=\begin{pmatrix} 0 & ① & 0 & ① & 0 & ① & ① \\ 0 & ② & ① & 0 & ① & 0 & ② \\ ① & 0 & 0 & ② & ② & 0 & ③ \\ ② & 0 & ② & 0 & 0 & ② & ④ \\ 0 & 0 & ③ & ③ & ③ & ③ & 0 \\ ③ & ③ & 0 & 0 & ④ & ④ & 0 \\ ④ & ④ & ④ & ④ & 0 & 0 & 0 \end{pmatrix},$$

where ① is the 1st row vector of incidence matrix of the BTD, ② is the 2nd row vector of incidence matrix of the BTD, ③ is the 3rd row vector of incidence matrix of the BTD, ④ is the 4th row vector of incidence matrix of the BTD, and 0 is a zero row vector of size B_2^*.

Now, we present another method for constructing BTD and will prove this BTD satisfies the property of nestedness. Consider a BIBD with parameters $v_1, b_1, r_1, k_1, \lambda_1$. We now construct another BIBD such that the number of treatments equals k_1, the block size of first BIBD. By using the treatment labels in the i[th] block of the first BIBD(for i= $1,2,\ldots,b_1$). we can get a BIBD for each i, with parameters $v_2=k_1, b_2, r_2, k_2, \lambda_2$. Here, we notice that the number of treatments in the second BIBD should be equal to the block size of the first BIBD. Now, append the ith block of the first BIBD to each of the blocks of the second BIBD. The resulting design is a nested BTD.

Theorem. If there exist a BIBD(D(1)) with parameters $v_1, b_1, r_1, k_1, \lambda_1$, and a BIBD(D(2)) with parameters $v_2=k_1, b_2, r_2, k_2, \lambda_2$, then there exists a BTD with parameters $V^*=v_1$, $B^*=b_1\times b_2$, $R^*=r_1(b_2+r_2)$, $K^*=k_1+k_2$, $\Lambda^*=(2r_2+b_2+\lambda_2)\lambda_1$

proof. Let the incidence matrix of D(1) be N_1. That is, $N_1=[\ n_{11}\ \ n_{12}\ \ldots\ n_{1b_1}\ ]$, where $n_{ij}=(n_{11j}\ n_{12j}\ldots n_{1v_1j})'$, $j=1,2,3,\ldots,b_1$. And let the incidence matrix of D(2) be N_2 using the treatment labels in the first block of D(1). Finally define a $v_1\times k_1$ matrix

$$d_l=\{d_{lus}\},\ l=1,2,\ldots,b_1,\ u=1,2,\ldots,v_1,\ s=1,2,\ldots,k_1.$$

The matrix $\{d_{lus}\}$ is a modified matrix from identify matrix such that u[th] row vector of matrix $\{d_{lus}\}$ equals zero vector if $n_{lus}=0$. Then, the incidence matrix of BTD is given by

$$N=[d_1N_2+n_{11}1'_{b_2},\ d_2N_2+n_{12}1'_{b_2},\dots,d_{b_1}N_2+n_{1b_1}1'_{b_2}]$$

$$=\sum_{j=1}^{b_1}{}^*(d_jN_2+n_{1j}1'_{b_2})\quad \text{, where } \sum{}^* \text{ means juxtaposition}$$

$$=[d_1N_2,\ d_2N_2,\dots,d_{b_1}N_2]+N_1\otimes 1'_{b_2}\,.$$

The concurrence matrix is

$$NN'=\sum_{j=1}^{b_1}(d_jN_2+n_{1j}1'_{b_2})(N_2'd_j'+1_{b_2}n'_{1j})$$

Since $N_2 1_{b_2}=r_2 1_{k_1}$, $d_j 1_{k_1}=n_{1j}$ for $j=1,2,\dots,b_1$, $\sum_{j=1}^{b_1}n_{1j}n'_{1j}=N_1N_1'=(r_1-\lambda_1)I_v+\lambda_1 1_v 1'_v$

$N_2N_2'=(r_2-\lambda_2)I_{k_1}+\lambda_2 1_{k_1}1'_{k_1}$, $\sum_{j=1}^{b_1}d_jd_j'=r_1I_v$, and $\sum_{j=1}^{b_1}d_j1_{k_1}1'_{k_1}d_j'=\sum_{j=1}^{b_1}n_{1j}n'_{1j}$,

we have

$$NN'=\{(r_1r_2-\lambda_1\lambda_2)+(2r_2+b_2)(r_1-\lambda_1)\}I_v+\{2r_2+b_2+\lambda_2)\lambda_1\}1_v1_v'$$

Here Λ^* is constant $((2r_2+b_2+\lambda_2)\lambda_1)$. This means a BTD. Notice that the structure of nestedness is $\left[\sum_{j=1}^{b_1}{}^*\ d_jN_2\ :\ N_1\otimes 1'_{b_2}\right]$, where $\sum{}^*$ means juxtaposition.

Example 6. Consider the following BIBD with parameters $v_1=b_1=4,\ r_1=k_1=3,\ \lambda_1=2$:

blocks	contents
1	1 2 3
2	1 2 4
3	1 3 4
4	2 3 4

$$,\quad N_1=\begin{pmatrix}1&1&1&0\\1&1&0&1\\1&0&1&1\\0&1&1&1\end{pmatrix}.$$

Consider the following another BIBD with parameters $v_2=k_1=b_2=3,\ r_2=k_2=2,\ \lambda_2=1$:

blocks	contents
1	1 2
2	1 3
3	2 3

$$,\quad N_2=\begin{pmatrix}1&1&0\\1&0&1\\0&1&1\end{pmatrix}.$$

We now construct nested BTD.

for $j=1$

$$d_1=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\\0&0&0\end{pmatrix},\quad d_1N_2=\begin{pmatrix}1&1&0\\1&0&1\\0&1&1\\0&0&0\end{pmatrix}.$$

for $j=2$

$$d_2=\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\\0&0&1\end{pmatrix},\quad d_2N_2=\begin{pmatrix}1&1&0\\1&0&1\\0&0&0\\0&1&1\end{pmatrix}.$$

for $j=3$

$$d_3=\begin{pmatrix}1&0&0\\0&0&0\\0&1&0\\0&0&1\end{pmatrix},\quad d_3N_2=\begin{pmatrix}1&1&0\\0&0&0\\1&0&1\\0&1&1\end{pmatrix}.$$

for $j=4$

$$d_4=\begin{pmatrix}0&0&0\\1&0&0\\0&1&0\\0&0&1\end{pmatrix},\quad d_4N_2=\begin{pmatrix}0&0&0\\1&1&0\\1&0&1\\0&1&1\end{pmatrix}.$$

$$N=[\,d_1N_2\;\; d_2N_2 \ldots d_{b_1}N_2]+N_1\otimes 1'_{b_2}$$

$$=\left\{\begin{pmatrix}1&1&0\\1&0&1\\0&1&1\\0&0&0\end{pmatrix}\begin{pmatrix}1&1&0\\1&0&1\\0&0&0\\0&1&1\end{pmatrix}\begin{pmatrix}1&1&0\\0&0&0\\1&0&1\\0&1&1\end{pmatrix}\begin{pmatrix}0&0&0\\1&1&0\\1&0&1\\0&1&1\end{pmatrix}\right\}+\begin{pmatrix}1&1&1&0\\1&1&0&1\\1&0&1&1\\0&1&1&1\end{pmatrix}\otimes(1\;\;1\;\;1)$$

$$=\begin{pmatrix}1&1&0&1&1&0&1&1&0&0&0&0\\1&0&1&1&0&1&0&0&0&1&1&0\\0&1&1&0&0&0&1&0&1&1&0&1\\0&0&0&1&1&0&1&1&0&1&1\end{pmatrix}+\begin{pmatrix}1&1&1&1&1&1&1&1&1&0&0&0\\1&1&1&1&1&1&0&0&0&1&1&1\\1&1&1&0&0&0&1&1&1&1&1&1\\0&0&0&1&1&1&1&1&1&1&1&1\end{pmatrix}$$

$$=\begin{pmatrix}2&2&1&2&2&1&2&2&1&0&0&0\\2&1&2&2&1&2&0&0&0&2&2&1\\1&2&2&0&0&0&2&1&2&2&1&2\\0&0&0&1&2&2&1&2&2&1&2&2\end{pmatrix}.$$

Allocation is as follows :

blocks	contents
1	1 1 2 2 3
2	1 2 2 3 3
3	1 2 2 3 3
4	1 1 2 2 4
5	1 1 2 4 4
6	1 2 2 4 4
7	1 1 3 3 4
8	1 1 3 4 4
9	1 3 3 4 4
10	2 2 3 3 4
11	2 2 3 4 4
12	2 3 3 4 4

Each block of this design can be sub-divided into two blocks of size 2 and 3 each, and let the contains of sub-blocks be as follows :

1	1	2	1	1	2	1	1	3	2	2	3	
2	3	3	2	4	4	3	4	4	3	4	4	→ M_1
1	1	1	1	1	1	1	1	1	2	2	2	
2	2	2	2	2	2	3	3	3	3	3	3	
3	3	3	4	4	4	4	4	4	4	4	4	→ M_2 .

(columns are blocks)

Let M_1 and M_2 be designs obtained by using sub-blocks. Let C_1 and C_2 be C-matrix of sub-designs. Then,

$$C_1=4(\ I+\frac{1}{2}J\ ) \text{ and } C_2=3(\ I+2J\ ).$$

These sub-designs are variance balanced.

Also, the super-block design is a variance balanced ternary design($n=3$). It is clear that

$$NN'=\begin{pmatrix} 27 & 16 & 16 & 16 \\ 16 & 27 & 16 & 16 \\ 16 & 16 & 27 & 16 \\ 16 & 16 & 16 & 27 \end{pmatrix},$$

$$C=rI-\frac{1}{k}NN'=12(\ I-\frac{1}{5}J\ )=11(I+\frac{16}{11}J).$$

Remarks. There is an easy way to construct a nested BTD. Construct a design D(2) using the treatments in the jth block of D(1) and append the jth block of D(1) to each block of D(2). We have just conducted, $j=1,2,3,4$

1) using 1st block of D(1), i. e. for $j=1$, we get 3 blocks (1 2), (1 3), (2 3) we append the 1st block of D(1) to each of these blocks to get the 3 blocks (1 2 1 2 3), (1 3 1 2 3), (2 3 1 2 3).

2) using 2nd block of D(1), i. e. for $j=2$, we get 3 blocks (1 2), (1 4), (2 4) we append the 2nd block of D(1) to each of these blocks to get the 3 blocks (1 2 1 2 4), (1 4 1 2 4), (2 4 1 2 4).

3) similarly for $j=3$ we get the 3 blocks (1 3 1 3 4), (1 4 1 3 4), (3 4 1 3 4).
4) and for $j=4$ we get the 3 blocks (2 3 2 3 4), (2 4 2 3 4), (3 4 2 3 4).

From 1), 2), 3) and 4), we can get a nested BTD which consist of 12 blocks.

4. Discussion

The problem of constructing nested balanced ternary designs (generally nested balanced n-ary designs) is not easy. The method we have provided in this paper may give us designs that require a large number of experimental units. The methods capable of constructing small sized balanced n-ary designs are certainly needed and should be developed in the future.

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