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Superoir and Inferior Limits on the Increments of Gaussian Processes†

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ABSTRACT

Csörgö-Révész type theorems for Wiener process are developed to those for Gaussian process. In particular, some results of superior and inferior limits for the increments of a Gaussian process are differently obtained under mild conditions, via estimating probability inequalities on the suprema of a Gaussian process.

Key Words : Wiener process; Gaussian process; Law of large numbers; Law of iterated logarithm; Regularly varying function and Borel-Cantelli lemma.

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1. INTRODUCTION AND RESULTS

Let $a_T (0 < T < \infty)$ be a monotonically nondecreasing function of T for which

(i) $0 < a_T \le T$,
(ii) T/a_T is monotonically nondecreasing.

For the standard Wiener process $\{W(t); 0 \le t < \infty\}$, Csörgö and Révész (1981) proved the following

Theorem A. If the function a_T satisfies the conditions (i) and (ii), then

$$\limsup_{T\to\infty} \sup_{0\le t\le T-a_T} \beta_T\, |W(t+a_T) - W(t)| = 1 \qquad \text{a.s.}$$

and

$$\limsup_{T\to\infty} \sup_{0\le t\le T-a_T} \sup_{0\le s\le a_T} \beta_T\, |W(t+s) - W(t)| = 1 \qquad \text{a.s.}$$

where $\beta_T = (2a_T(\log(T/a_T) + \log\log T))^{-1/2}$. If we also have the condition

(iii) $\lim_{T\to\infty} \log(T/a_T)/\log\log T = \infty$,

then

$$\lim_{T\to\infty} \sup_{0\le t\le T-a_T} \beta_T\, |W(t+a_T) - W(t)| = 1 \qquad \text{a.s.}$$

and

$$\lim_{T\to\infty} \sup_{0\le t\le T-a_T} \sup_{0\le s\le a_T} \beta_T\, |W(t+s) - W(t)| = 1 \qquad \text{a.s.}$$

From Theorem A, we obtain the following Erdös-Rényi's (1970) law of large numbers when $a_T = c\log T$, $c > 0$:

$$\lim_{T\to\infty} \sup_{0\le t\le T-c\log T} \frac{|W(t+c\log T) - W(t)|}{\sqrt{2c}\log T} = 1 \qquad \text{a.s.}$$

It is well known that the Strassen's (1964) law of iterated logarithm implies that, for any c with $0 < c \le 1$,

$$\limsup_{T\to\infty} \sup_{0\le t\le T-cT} \frac{|W(t+cT) - W(t)|}{\sqrt{2cT\log\log T}} = 1 \qquad \text{a.s.} \tag{1.1}$$

It should be observed that $a_T = cT$, $0 < c \le 1$, fails to satisfy the condition (iii). Hence, by Theorem A, we can not have a "limit" value as in (1.1). In fact, Deo (1977) showed that the "lim inf" differs from the "lim sup" if the condition (iii) of Theorem A is replaced by

$$\lim_{T\to\infty} \log(T/a_T)/\log\log T < \infty.$$

Furthermore, Book and Shore (1978) obtained the following result under the wider condition (iv) as below than (iii):

Theorem B. Let the function a_T satisfy the conditions (i), (ii) and

(iv) $\lim_{T\to\infty} \log(T/a_T)/\log\log T = r, \qquad 0 \le r \le \infty,$

then

$$\limsup_{T\to\infty} \sup_{0\le t\le T-a_T} \beta_T\, |W(t+a_T) - W(t)| = 1 \qquad \text{a.s.} \tag{1.2}$$

and

$$\liminf_{T\to\infty} \sup_{0\le t\le T-a_T} \beta_T\, |W(t+a_T) - W(t)| = \sqrt{\frac{r}{r+1}} \qquad \text{a.s.} \tag{1.3}$$

where $\beta_T = (2a_T(\log(T/a_T) + \log\log T))^{-1/2}$.

Csáki et al. (1991), Ortega (1984) and Choi (1991) extended Theorem A for Wiener process to a Gaussian process as in the next Theorem C. Let $\{X(t) : 0 \le t < \infty\}$ be an almost surely continuous Gaussian process with $X(0) = 0$, $E\{X(t)\} = 0$ and stationary increments $E\{X(t) - X(s)\}^2 = \sigma^2(|t-s|)$, where $\sigma(y)$ is a function of $y \ge 0$. Further we assume that $\sigma(t)$, $t > 0$, is a nondecreasing continuous, regularly varying function with exponent λ $(0 < \lambda < 1)$ at infinity (e.g., if $\{X(t) : 0 \le t < \infty\}$ is a standard Wiener process, then $\sigma(t) = \sqrt{t}$, $t > 0$). Let a_T $(0 < T < \infty)$ be a monotonically nondecreasing function of T for which

(i) $0 < a_T \le T$,
(ii) T/a_T is monotonically nondecreasing.

Define the incremental processes $H_0(T, a_T), \dots, H_4(T, a_T)$ by

$$\begin{aligned}
H_0(T,a_T) &= |X(T+a_T) - X(T)|,\\
H_1(T,a_T) &= \sup_{0\le s\le a_T} |X(T+s) - X(T)|,\\
H_2(T,a_T) &= \sup_{0\le t\le T-a_T} |X(t+a_T) - X(t)|,\\
H_3(T,a_T) &= \sup_{0\le s\le a_T}\ \sup_{0\le t\le T-a_T} |X(t+s) - X(t)|,
\end{aligned}$$

$$H_4(T, a_T) = \sup_{0<u-v\leq a_T} \sup_{0\leq v<u\leq T} |X(u) - X(v)|,$$

respectively, and for $T > e$ put

$$\alpha_T = \{2\sigma^2(a_T)(\log(T/a_T) + \log\log T)\}^{-1/2}$$

Theorem C. (Csáki et al. (1991), Ortega (1984) and Choi (1991)) Let $\{X(t) : 0 \leq t < \infty\}$ be a centered Gaussian process such that $X(0) = 0$ and $E\{X(t) - X(s)\}^2 = \sigma^2(|t-s|)$. Assume that, for $t > 0$, $\sigma(t) = t^\lambda$, $0 < \lambda < 1$. Let a_T $(0 < T < \infty)$ be a monotonically nondecreasing function of T for which

(i) $0 < a_T \leq T$,

(ii) T/a_T is monotonically nondecreasing.

Then

$$\limsup_{T\to\infty} \alpha_T H_i(T, a_T) = 1 \qquad i = 0, 1, 2, 3, 4 \qquad \text{a.s.} \tag{1.4}$$

where $\alpha_T = \{2\sigma^2(a_T)(\log(T/a_T) + \log\log T)\}^{-1/2}$. If we also have the condition

(iii) $\lim_{T\to\infty} \log(T/a_T)/\log\log T = \infty$,

then

$$\lim_{T\to\infty} \alpha_T H_i(T, a_T) = 1 \qquad i = 2, 3, 4 \qquad \text{a.s.} \tag{1.5}$$

Let a_T $(0 < T < \infty)$ satisfy the above conditions (i) and (ii). For our purpose of this paper, we define $H_5(T, a_T)$ and $H_6(T, a_T)$ by

$$H_5(T, a_T) = \sup_{0\leq s\leq a_T} \sup_{0\leq t\leq T-s} |X(t+s) - X(t)|,$$
$$H_6(T, a_T) = \sup_{0\leq s\leq a_T} \sup_{0\leq t\leq T} |X(t+s) - X(t)|,$$

respectively. Note that the region under two sups in $H_4(T, a_T)$ has a larger area $a_T^2/2$ than that of $H_3(T, a_T)$, and that the $H_6(T, a_T)$ case also has a larger area $a_T^2/2$ than that of $H_4(T, a_T)$. For $T > e$, denote γ_T by

$$\gamma_T = \{2\sigma^2(a_T)(\log(T/a_T) + \delta\log\log T)\}^{-1/2}$$

where δ is given in $0 < \delta < \infty$.

The main objective of this paper is, under the condition (iv) of Theorem B, to develope Theorems A, B and C to Gaussian processes for $H_i(T, a_T)$, $i = 0, 1, 2, 3, 4, 5, 6$, and to show that, for $H_i(T, a_T)$, $i = 0, 1, 2, 3, 4, 5, 6$, the superior limit differs from the inferior limit when $r < \infty$ in the condition (iv). Our results are as follows:

Theorem 1. Let $\{X(t) : 0 \leq t < \infty\}$ be a centered Gaussian process such that $X(0) = 0$ and $E\{X(t) - X(s)\}^2 = \sigma^2(|t - s|)$. Assume that, for $t > 0$, $\sigma(t) = t^\lambda$, $0 < \lambda < 1$. Let a_T $(0 < T < \infty)$ be a monotonically nondecreasing function of T for which

(i) $0 < a_T \leq T$,
(ii) T/a_T is monotonically nondecreasing,
(iv) $\lim_{T\to\infty} \log(T/a_T)/\log\log T = r, \qquad 0 \leq r \leq \infty.$

Then we have

$$\limsup_{T\to\infty} \gamma_T H_i(T, a_T) = \sqrt{\frac{r+1}{r+\delta}} \qquad i = 0, 1, 2, 3, 4, 5, 6 \qquad \text{a.s.} \tag{1.6}$$

where $\gamma_T = \{2\sigma^2(a_T)(\log(T/a_T) + \delta \log\log T)\}^{-1/2}$ and δ is given in $0 < \delta < \infty$.

Theorem 2. Let $\{X(t) : 0 \leq t < \infty\}$ be a centered Gaussian process such that $X(0) = 0$ and $E\{X(t) - X(s)\}^2 = \sigma^2(|t - s|)$. Assume that, for $t > 0$, $\sigma(t) = t^\lambda$, $0 < \lambda < 1$. Let a_T $(0 < T < \infty)$ be a monotonically nondecreasing function of T for which

(i) $0 < a_T \leq T$,
(ii) T/a_T is monotonically nondecreasing,
(iv) $\lim_{T\to\infty} \log(T/a_T)/\log\log T = r, \qquad 0 \leq r \leq \infty.$

Then we have

$$\liminf_{T\to\infty} \gamma_T \; H_i(T, a_T) = \sqrt{\frac{r}{r+\delta}} \qquad i = 2, 3, 4, 5, 6 \qquad \text{a.s.} \tag{1.7}$$

Remark. Comparing our theorems with Theorem C, the results (1.6) and (1.7) still hold true for $i = 5, 6$. For $H_i(T, a_T)$, $i = 2, 3, 4, 5, 6$, the superior and inferior limits in Theorems 1 and 2 are evidently different when $0 \leq r < \infty$. If $\delta = 1$ and $\sigma(t) = \sqrt{t}$ for $t > 0$ in Theorems 1 and 2, then we obtain the results (1.2) and (1.3) of Theorem B. Also if $\delta = 1$ and $r = \infty$, then Theorems 1 and 2 yield the result (1.5) of Theorem C.

Example. If $a_T = T/(\log T - \log\log T)^2$ for $T > e$, then $r = 2$ by (iv). Thus by (1.6)

$$\limsup_{T\to\infty} \gamma_T H_i(T, a_T) = \sqrt{\frac{3}{2+\delta}}, \qquad i = 0,1,2,3,4,5,6 \qquad \text{a.s.}$$

and by (1.7)

$$\liminf_{T\to\infty} \gamma_T H_i(T, a_T) = \sqrt{\frac{2}{2+\delta}}, \qquad i = 2,3,4,5,6 \qquad \text{a.s.}$$

where δ $(0 < \delta < \infty)$ is a constant given in γ_T.

2. PROOFS

In order to prove Theorems 1 and 2, we need some lemmas: the following Lemma 1 is a well-known Slepian's lemma.

Lemma 1.(Slepian (1962)) Suppose that $\{U_i : i = 1, 2, \cdot, n\}$ and $\{V_i : i = 1, 2, \cdot, n\}$ are jointly standardized normal random variables with

$$\text{covariance}(U_i, U_j) \le \text{covariance}(V_i, V_j), \quad i \ne j.$$

Then for any real number u,

$$\mathbf{P}\{\max_{1\le i\le n} U_i \le u\} \le \mathbf{P}\{\max_{1\le i\le n} V_i \le u\}.$$

The following lemma is easily shown by the same method as the proof in Lemma 4.5 of Choi (1991):

Lemma 2. Let $\{X(t) : 0 \le t < \infty\}$ be an almost surely continuous Gaussian process such that $X(0) = 0$, $E\{X(t)\} = 0$ and $E\{X(t) - X(s)\}^2 = \sigma^2(|t-s|)$. Assume that $\sigma(t), t > 0$, is a nondecreasing continuous, regularly varying function at ∞ with exponent λ for some $0 < \lambda < 1$. Let a_T $(0 < T < \infty)$ be a nondecresing function of T for which

(i) $0 < a_T \le T$,
(ii) T/a_T is nondecreasing.

Then for any small $\epsilon > 0$ there exist large constants $u_0 = u_0(T, \epsilon) > 0$ and C_ϵ depending only on ϵ such that for all $u \ge u_0$

$$\mathbf{P}\Big\{\sup_{0\le s\le a_T} \sup_{0\le t\le T} \frac{X(t+s) - X(t)}{\sigma(a_T)} \ge u\Big\} \le C_\epsilon\Big(\frac{T}{a_T}\Big)\, e^{-u^2/(2+\epsilon)}.$$

The next lemmas are essential to prove Theorem 2. The following Lemma 3 is easily obtained by a slight modification of the proof of Theorem 1 in Lu (1986):

Lemma 3. (Lu (1986)) Let $\{X(t) : 0 \le t < \infty\}$ be a centered Gaussian process such that $X(0) = 0$ and $E\{X(t) - X(s)\}^2 = \sigma^2(|t-s|)$. For $t > 0$, let $\sigma(t) = t^\lambda$, $0 < \lambda < 1$. Then

$$\lim_{h \downarrow 0} \sup_{0 \le v \le h} \sup_{0 \le u \le 1} \frac{|X(u+v) - X(u)|}{\sqrt{2\log(1/h)}\sigma(h)} = 1 \qquad \text{a.s.}$$

Lemma 4. Let m and k be arbitrary positive numbers. Let $\{X(t) : 0 \le t < \infty\}$ be a centered Gaussian process such that $X(0) = 0$ and $E\{X(t) - X(s)\}^2 = \sigma^2(|t-s|)$. For $t > 0$, let $\sigma(t) = t^\lambda$, $0 < \lambda \le 1/2$. Then for $u \ge 0$ we have

$$\mathbf{P}\Big\{\sup_{0 \le t \le k} \frac{X(t+m) - X(t)}{\sigma(m)} \le u\Big\} \le \exp\Big(-\frac{k}{m}\frac{1}{\sqrt{2\pi}(u+1)}e^{-u^2/2}\Big).$$

Proof. Set

$$\Phi(u) = \int_u^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,dx, \quad u \ge 0.$$

Then, by Fernique (1975, p.71), the inequality

$$\frac{1}{\sqrt{2\pi}(u+1)}e^{-u^2/2} \le \Phi(u) \le \frac{4}{3}\frac{1}{\sqrt{2\pi}(u+1)}e^{-u^2/2} \tag{2.1}$$

holds for all $u \ge 0$. For given k and m, let $h = [k/m]$, where $[y]$ denotes the greatest integer not exceeding y. For $i = 0, 1, \cdots, h$, define the incremental random variable

$$Z(i) = X((i+1)m) - X(im).$$

Since $\sigma^2(t) = t^{2\lambda}$, $0 < \lambda \le 1/2$, it follows from the elementary relation $ab = (a^2 + b^2 - (a-b)^2)/2$ that, for $l = |i-j| \ge 1$,

$$\begin{aligned} &covariance(Z(i),\ Z(j)) \\ &\quad = E\{X(im)X(jm)\} - E\{X(im)X((j-1)m)\} - E\{X((i-1)m)X(jm)\} \\ &\qquad + E\{X((i-1)m)X((j-1)m)\} \end{aligned}$$

$$= \frac{1}{2}\{(\sigma^2((l+1)m) - \sigma^2(lm)) - (\sigma^2(lm) - \sigma^2((l-1)m)\} \leq 0. \tag{2.2}$$

In order to apply Lemma 1, let $U_i = Z(i)/\sigma(m)$ in Lemma 1 and V_i be independent standard normal random variables. From (2.2), $covariance(U_i, U_j) \leq covariance(V_i, V_j) = 0, \quad i \neq j$. Applying the inequality (2.1) and Lemma 1, we have

$$\begin{aligned}
&\mathbf{P}\Big\{\sup_{0\leq t\leq k} \frac{X(t+m)-X(t)}{\sigma(m)} \leq u\Big\} \leq \mathbf{P}\Big\{\max_{0\leq i\leq h} U_i \leq u\Big\} \\
&\leq \mathbf{P}\Big\{\max_{0\leq i\leq h} V_i \leq u\Big\} = \mathbf{P}\Big\{V_0 \leq u, \ldots, V_h \leq u\Big\} \\
&= \{1-\Phi(u)\}^{h+1} \leq \exp(-(h+1)\Phi(u)) \\
&\leq \exp\Big(-\frac{h+1}{\sqrt{2\pi}(u+1)}e^{-u^2/2}\Big) \leq \exp\Big(-\frac{k}{m}\frac{1}{\sqrt{2\pi}(u+1)}e^{-u^2/2}\Big).
\end{aligned}$$

Lemma 5. (Berman (1964)) Let $\{X_i : i = 1, 2, \ldots, n\}$ be jointly standardized normal random variables with covariance$(X_i, X_j) = \Lambda_{ij}$ such that

$$\rho = \max_{i\neq j} |\Lambda_{ij}| < 1.$$

Then, for any real number u and integers $1 \leq l_1 < l_2 < \ldots < l_k \leq n$ with $k \leq n$,

$$\mathbf{P}\Big\{\max_{1\leq i\leq k} X_{l_i} \leq u\Big\} \leq \Big\{1-\Phi(u)\Big\}^k + K \sum_{1\leq i<j\leq k} |r_{ij}| \exp\Big(-\frac{u^2}{1+|r_{ij}|}\Big) \tag{2.3}$$

where $r_{ij} = \Lambda_{l_i l_j}$ and $K = K(\rho)$ is a positive constant depending on ρ but not n, u and k.

Lemma 6. (Choi (1991), page 204) Let $X(t), \sigma(t)$ and a_T be as in Lemma 2. For $0 < \alpha < 1$, set $T_k = \exp(k^\alpha),\ k = 1, 2, \cdots$, and let T be in $T_k \leq T \leq T_{k+1}$. Then we have

$$\begin{aligned}
&\liminf_{T\to\infty} \gamma_T \sup_{0\leq t\leq T-a_T} (X(t+a_T) - X(t)) \\
&\geq \liminf_{k\to\infty} \gamma_{T_k} \sup_{0\leq t\leq T_k - a_{T_k}} (X(t+a_{T_k}) - X(t) \qquad \text{a.s.}
\end{aligned}$$

We are now ready to prove our theorems. In the proofs we shall let c denote a positive constant changing in lines if necessary.

Proof of Theorem 1. If $i = 0,1,2,3,4$, then it is obvious from (1.4) of Theorem C that

$$\limsup_{T\to\infty} \gamma_T H_i(T, a_T) = \sqrt{\frac{r+1}{r+\delta}} \limsup_{T\to\infty} \alpha_T H_i(T, a_T) = \sqrt{\frac{r+1}{r+\delta}} \quad \text{a.s.}$$

Since $H_6(T, a_T)$ is the largest random variable of all $H_i(T, a_T)$, $i = 0,1,2,3,4,5,6$, Theorem 1 immediately follows only if we show that

$$\limsup_{T\to\infty} \gamma_T H_6(T, a_T) \le \sqrt{\frac{r+1}{r+\delta}} \quad \text{a.s.} \tag{2.4}$$

First suppose that $0 \le r < \infty$. Applying the condition (iv) and Lemma 2, we have, for any small $\epsilon > 0$

$$\begin{aligned}
&\mathbf{P}\Big\{\gamma_T H_6(T, a_T) > \sqrt{\Big(\frac{r+1}{r+\delta}\Big)\Big(\frac{1+\epsilon}{1-\epsilon}\Big)}\Big\} \\
&= \mathbf{P}\Big\{\sup_{0\le s\le a_T} \sup_{0\le t\le T} |X(t+s) - X(t)| > \sqrt{\Big(\frac{r+1}{r+\delta}\Big)\Big(\frac{1+\epsilon}{1-\epsilon}\Big)}\gamma_T^{-1}\Big\} \\
&\le 2\mathbf{P}\Big\{\sup_{0\le s\le a_T} \sup_{0\le t\le T} \frac{X(t+s) - X(t)}{\sigma(a_T)} \\
&\qquad > \sqrt{\Big(\frac{r+1}{r+\delta}\Big)\Big(\frac{1+\epsilon}{1-\epsilon}\Big)}\Big(2\log\Big(\frac{T}{a_T}(\log T)^\delta\Big)\Big)^{1/2}\Big\} \\
&\le C_\epsilon \frac{T}{a_T} \exp\Big\{-\frac{1}{2+\epsilon}\Big(\frac{r+1}{r+\delta}\Big)\Big(\frac{1+\epsilon}{1-\epsilon}\Big) 2\log\Big(\frac{T}{a_T}(\log T)^\delta\Big)\Big\} \\
&= C_\epsilon \frac{T}{a_T}\Big\{\frac{T}{a_T}(\log T)^\delta\Big\}^{-\frac{2+2\epsilon}{(2+\epsilon)(1-\epsilon)}\left(\frac{r+1}{r+\delta}\right)} \\
&\le C_\epsilon (\log T)^{r - \frac{(2+2\epsilon)(r+1)}{(2+\epsilon)(1-\epsilon)}} = C_\epsilon (\log T)^{\frac{-2-2\epsilon-\epsilon(3r+r\epsilon)}{(2+\epsilon)(1-\epsilon)}}
\end{aligned} \tag{2.5}$$

provided $T > 0$ is big enough. For given positive integer k, let $T_k = \exp(k^{1-\epsilon})$ for any small $\epsilon > 0$. Then the inequality (2.5) implies that

$$\mathbf{P}\Big\{\gamma_{T_k} H_6(T_k, a_{T_k}) > \sqrt{\Big(\frac{r+1}{r+\delta}\Big)\Big(\frac{1+\epsilon}{1-\epsilon}\Big)}\Big\} \le C_\epsilon \; k^{-1-\{\epsilon(1+3r+r\epsilon)/(2+\epsilon)\}}\Big\}$$

and hence

$$\sum_{k=1}^{\infty} \mathbf{P}\Big\{\gamma_{T_k} H_6(T_k, a_{T_k}) > \sqrt{\Big(\frac{r+1}{r+\delta}\Big)\Big(\frac{1+\epsilon}{1-\epsilon}\Big)}\Big\} < \infty.$$

Thus, by the Borel-Cantelli lemma, we obtain

$$\limsup_{k\to\infty} \gamma_{T_k} H_6(T_k, a_{T_k}) \le \sqrt{\frac{r+1}{r+\delta}} \qquad \text{a.s.} \tag{2.6}$$

when $0 \le r < \infty$.

Next, let $r = \infty$. In this case, for given $0 < \delta < \infty$, we shall regard $\sqrt{(r+1)/(r+\delta)}$ as 1. From (iv), the fact that $r = \infty$ implies that, for any small $\epsilon > 0$,

$$\frac{T}{a_T} > (\log T)^{1/\epsilon}$$

holds as long as T is big enough. By this and Lemma 2, we have, for any small $\epsilon > 0$

$$\begin{aligned}
&\mathbf{P}\Big\{\gamma_T H_6(T, a_T) > \sqrt{\frac{1+\epsilon}{1-\epsilon}}\Big\} \\
&\le C_\epsilon \frac{T}{a_T} \exp\Big(-\frac{1}{2+\epsilon}\Big(\frac{1+\epsilon}{1-\epsilon}\Big) 2\log\Big(\frac{T}{a_T}(\log T)^\delta\Big)\Big) \\
&= C_\epsilon \Big(\frac{T}{a_T}\Big)^{(-3\epsilon-\epsilon^2)/\{(2+\epsilon)(1-\epsilon)\}} (\log T)^{-\delta(2+2\epsilon)/\{(2+\epsilon)(1-\epsilon)\}} \\
&\le C_\epsilon (\log T)^{(-3-\epsilon)/\{(2+\epsilon)(1-\epsilon)\} - [\delta(2+2\epsilon)/\{(2+\epsilon)(1-\epsilon)\}]} \\
&= C_\epsilon (\log T)^{-(3+2\delta+\epsilon+2\delta\epsilon)/\{(2+\epsilon)(1-\epsilon)\}}
\end{aligned} \tag{2.7}$$

provided T is big enough. Let $T_k = \exp(k^{1-\epsilon})$ for any small $\epsilon > 0$. Then (2.7) implies

$$\mathbf{P}\Big\{\gamma_{T_k} H_6(T_k, a_{T_k}) > \sqrt{\frac{1+\epsilon}{1-\epsilon}}\Big\} \le C_\epsilon k^{-(3+2\delta+\epsilon+2\delta\epsilon)/(2+\epsilon)}$$

and hence we obtain

$$\limsup_{k\to\infty} \gamma_{T_k} H_6(T_k, a_{T_k}) \le 1 \qquad \text{a.s} \tag{2.8}$$

when $r = \infty$. Let T be in $T_{k-1} \le T \le T_k$ for each positive integer k. Considering (2.6) and (2.8), the remainder of the proof of (2.4) is to show that

$$\limsup_{T\to\infty} \gamma_T H_6(T, a_T) \le \limsup_{k\to\infty} \gamma_{T_k} H_6(T_k, a_{T_k}). \tag{2.9}$$

Clearly, the inequality

$$\gamma_{T_k} H_6(T_k, a_{T_k}) \ge \gamma_{T_k} \sup_{0\le s\le a_T} \sup_{0\le t\le T} |X(t+s) - X(t)| \ge \frac{\gamma_{T_k}}{\gamma_{T_{k-1}}} \gamma_T H_6(t, a_T) \tag{2.10}$$

holds. By the conditions (i) and (ii), we get

$$\frac{a_{T_k}}{a_{T_{k-1}}} \le \frac{T_k}{T_{k-1}} \le \exp((1-\epsilon)(k-1)^{-\epsilon})$$

where the last inequality is easily shown by the mean-value theorem. Since $\sigma(\cdot)$ is nondecreasing continuous, regularly varying at ∞ and $(\log u)/u$ is decreasing for $u > e$, it follows from (ii) that

$$\begin{aligned}
1 &\ge \frac{\gamma_{T_k}}{\gamma_{T_{k-1}}} \ge \frac{\sigma(a_{T_{k-1}})}{\sigma(a_{T_k})}\Big(\frac{(\log T_{k-1})^\delta T_{k-1}/a_{T_{k-1}}}{(\log T_k)^\delta T_k/a_{T_k}}\Big)^{1/2} \\
&\ge \frac{\sigma(\exp(-(1-\epsilon)(k-1)^{-\epsilon})a_{T_k})}{\sigma(a_{T_k})}\Big(\frac{(k-1)^{\delta(1-\epsilon)}T_{k-1}}{k^{\delta(1-\epsilon)}T_k}\Big)^{1/2} \\
&\ge \exp\{-(1-\epsilon)(k-1)^{-\epsilon})(\lambda+\eta)\} \\
&\qquad \times \exp\{-\frac{1}{2}(1-\epsilon)(k-1)^{-\epsilon}\}\Big(\frac{k-1}{k}\Big)^{\delta(1-\epsilon)/2} \\
&\longrightarrow 1 \quad \text{as } k \to \infty,
\end{aligned}$$

where $\eta > 0$ is small enough. This proves (2.9) by (2.10), and hence (2.4) follows.

Proof of Theorem 2. We shall first prove that for $0 \le r \le \infty$

$$\liminf_{T\to\infty} \gamma_T \; H_6(T, a_T) \le \sqrt{\frac{r}{r+\delta}} \quad \text{a.s.} \tag{2.11}$$

Let $0 < r \le \infty$. By (iv), this case does not imply $a_T/T \to \beta$ for some $0 < \beta \le 1$ but $a_T/T \to 0$, as $T \to \infty$. Set

$$A_T = \sqrt{1 + \{(\delta \log\log T)/\log(T/a_T)\}}.$$

Then $A_T \to \sqrt{(r+\delta)/r}$ as $T \to \infty$ by (iv). Let us denote $U(t) \stackrel{d}{=} V(t)$ if the process $U(t)$ has the same distribution as the process $V(t)$. Since $\sigma(t) = t^\lambda$ $(0 < \lambda < 1)$, $\sigma(T)X(t/T) \stackrel{d}{=} X(t)$. By this change of scale, as in page 5 of Book and Shore (1978), we have

$$\begin{aligned}
\gamma_T H_6(T, a_T) &= \sup_{0\le s\le a_T} \; \sup_{0\le t\le T} \frac{|X(t+s) - X(t)|}{\sqrt{2\log(T/a_T)}A_T\sigma(a_T)} \\
&\stackrel{d}{=} \sup_{0\le s/T\le a_T/T} \; \sup_{0\le t/T\le 1} \frac{\sigma(T)|X((t+s)/T) - X(t/T)|}{\sqrt{2\log(T/a_T)}A_T\sigma(a_T)} := B_6(T, a_T)
\end{aligned}$$

$$= \sup_{0 \le v \le h} \sup_{0 \le u \le 1} \frac{|X(u+v) - X(u)|}{\sqrt{2\log(1/h)}\sigma(h)A_T}. \tag{2.12}$$

Since we are in the case $h = a_T/T \to 0$ as $T \to \infty$, it follows from Lemma 3 that

$$\lim_{T\to\infty} B_6(T, a_T) = \sqrt{r/(r+\delta)} \qquad \text{a.s.}$$

This implies

$$\lim_{T\to\infty} \gamma_T H_6(T, a_T) = \sqrt{r/(r+\delta)} \qquad \text{in probability.}$$

Hence we can find a subsequence $\{T_k : 1 \le k < \infty\}$ such that

$$\lim_{k\to\infty} \gamma_{T_k} H_6(T_k, a_{T_k}) = \sqrt{r/(r+\delta)} \qquad \text{a.s.}$$

Therefore, we have

$$\liminf_{T\to\infty} \gamma_T H_6(T, a_T) \le \sqrt{r/(r+\delta)} \qquad \text{a.s.} \tag{2.13}$$

for $0 < r \le \infty$. Consider the case where $r = 0$ in (iv). Clearly,

$$\liminf_{T\to\infty} \gamma_T H_i(T, a_T) \ge 0, \qquad i = 0, 1, \ldots, 6 \qquad \text{a.s.} \tag{2.14}$$

If we define $1/0 = \infty$, then by (2.12) and Lemma 3

$$\lim_{T\to\infty} B_6(T, a_T) = 0 \qquad \text{a.s.}$$

By the same method as in the above statements, we can also deduce

$$\liminf_{T\to\infty} \gamma_T H_6(T, a_T) \le 0 \qquad \text{a.s.} \tag{2.15}$$

Thus by (2.13) and (2.15), the inequality (2.11) holds for $0 \le r \le \infty$. The inequalities (2.14) and (2.15) also lead to, for $r = 0$,

$$\liminf_{T\to\infty} \gamma_T H_i(T, a_T) = 0, \qquad i = 0, 1, \ldots, 6 \qquad \text{a.s.} \tag{2.16}$$

We next prove that for $0 \le r \le \infty$

$$\liminf_{T\to\infty} \gamma_T \; H_2(T, a_T) \ge \sqrt{\frac{r}{r+\delta}} \qquad \text{a.s.} \tag{2.17}$$

Recall that

$$\lim_{T\to\infty}\frac{\gamma_T}{\alpha_T}=\sqrt{\frac{r+1}{r+\delta}}.$$

If $r=\infty$, then γ_T is a symptotically the same as α_T as long as T is big enough. It follows from (1.5) of Theorem C that for $r=\infty$

$$\liminf_{T\to\infty}\gamma_T H_2(T,a_T)=\liminf_{T\to\infty}\alpha_T H_2(T,a_T)=1\geq\sqrt{\frac{r}{r+\delta}}\qquad\text{a.s.}$$

In what follows, we assume that $0<r<\infty$, because (2.16) holds when $r=0$.

Let $0<\lambda\leq 1/2$. Using Lemma 4 and the condition (iv), we have, for any small ϵ with $0<\epsilon<r$

$$\begin{aligned}
&P\Big\{\gamma_T\sup_{0\le t\le T-a_T}(X(t+a_T)-X(t))\leq\sqrt{\frac{r-\epsilon}{r+\delta}}(1-\epsilon)\Big\}\\
&=P\Big\{\sup_{0\le t\le T-a_T}\frac{X(t+a_T)-X(t)}{\sigma(a_T)}\\
&\qquad\qquad\leq\sqrt{\frac{r-\epsilon}{r+\delta}}(1-\epsilon)(2\log(\frac{T}{a_T}(\log T)^{\delta}))^{1/2}\Big\}\\
&\leq\exp\Big(-\frac{T-a_T}{a_T}\frac{1}{\sqrt{2\pi}}\Big\{\sqrt{\frac{r-\epsilon}{r+\delta}}(1-\epsilon)(2\log(\frac{T}{a_T}(\log T)^{\delta}))^{1/2}+1\Big\}^{-1}\\
&\qquad\qquad\times(\frac{T}{a_T}(\log T)^{\delta})^{-\frac{r-\epsilon}{r+\delta}(1-\epsilon)^2}\Big)\\
&\leq\exp\Big(-c(\frac{T}{a_T})^{1-\frac{r-\epsilon}{r+\delta}(1-\epsilon)}(\log T)^{-\frac{\delta(r-\epsilon)}{r+\delta}(1-\epsilon)}\Big)\\
&\leq\exp\Big(-c(\log T)^{r(1-\frac{r-\epsilon}{r+\delta}(1-\epsilon))-\frac{\delta(r-\epsilon)}{r+\delta}(1-\epsilon)}\Big)\\
&=\exp\Big(-c(\log T)^{r\epsilon+\epsilon(1-\epsilon)}\Big)
\end{aligned}$$

for all large $T>0$, where c is a positive constant. For $0<\alpha<1$, putting $T_k=\exp(k^{\alpha})$, $k=1,2,\cdots$, we have

$$\begin{aligned}
&\mathbf{P}\Big\{\gamma_{T_k}\sup_{0\le t\le T_k-a_{T_k}}(X(t+a_{T_k})-X(t))\leq\sqrt{\frac{r-\epsilon}{r+\delta}}(1-\epsilon)\Big\}\\
&\leq\exp(-ck^{\alpha(r\epsilon+\epsilon(1-\epsilon))}).
\end{aligned}$$

The Borel-Cantelli lemma implies that

$$\liminf_{k\to\infty}\gamma_{T_k}\sup_{0\le t\le T_k-a_{T_k}}(X(t+a_{T_k})-X(t))\geq\sqrt{\frac{r}{r+\delta}}\qquad\text{a.s.}$$

Let T be in $T_k \le T \le T_{k+1}$, $k = 1, 2, \cdots$, then (2.17) follows from Lemma 6.

Next assume that $1/2 < \lambda < 1$. If $T > 0$ is big enough, then by the condition (iv) we can choose a big number $M > 0$ such that

$$M < (\log T)^B < T/a_T \tag{2.18}$$

for some $B > 0$. Define a positive integer $k_T = [T/(Ma_T)]$. By (2.18), k_T is increasing on T. For $i = 1, 2, \cdots, k_T$, we define the incremental random variables

$$Y_T(i) = X(Mia_T) - X((Mi-1)a_T).$$

Then $Y_T(i)/\sigma(a_T)$ is a standard normal random variable. It follows that, for any small $\epsilon > 0$,

$$\begin{aligned}
&\mathbf{P}\Big\{\gamma_T \sup_{0\le t\le T-a_T} (X(t+a_T) - X(t)) \le \sqrt{\frac{r-\epsilon}{r+\delta}}(1-\epsilon)\Big\} \\
&= \mathbf{P}\Big\{\sup_{0\le t\le T-a_T} \frac{X(t+a_T)-X(t)}{\sigma(a_T)} \le \sqrt{\frac{r-\epsilon}{r+\delta}}(1-\epsilon)\Big(2\log(\frac{T}{a_T}(\log T)^\delta)\Big)^{1/2}\Big\} \\
&\le \mathbf{P}\Big\{\max_{1\le i\le k_T} \frac{Y_T(i)}{\sigma(a_T)} \le \sqrt{\frac{r-\epsilon}{r+\delta}}(1-\epsilon)\Big(2\log(\frac{T}{a_T}(\log T)^\delta)\Big)^{1/2}\Big\}
\end{aligned} \tag{2.19}$$

Let $r_T(i,j) = covariance(Y_T(i)/\sigma(a_T), Y_T(j)/\sigma(a_T))$, $i \ne j$, and let $m = i - j \ge 1$ without loss of generality. Using the elementary relation $ab = (a^2 + b^2 - (a-b)^2)/2$ and the mean-value theorem, we have

$$\begin{aligned}
&r_T(i,j) \\
&= E\{Y_T(i)Y_T(j)\}/\sigma^2(a_T) \\
&= E\{X(Mia_T)X(Mja_T) - X(Mia_T)X((Mj-1)a_T) \\
&\qquad -X((Mi-1)a_T)X(Mja_T) \\
&\qquad +X((Mi-1)a_T)X((Mj-1)a_T)\}/\sigma^2(a_T) \\
&= \frac{1}{2\sigma^2(a_T)}E\Big\{-(X(Mia_T) - X(Mja_T))^2 \\
&\qquad +(X(Mia_T) - X((Mj-1)a_T))^2 + (X((Mi-1)a_T) - X(Mja_T))^2 \\
&\qquad -(X((Mi-1)a_T) - X((Mj-1)a_T))^2\Big\} \\
&= \frac{1}{2\sigma^2(a_T)}\{-\sigma^2(mMa_T) + \sigma^2((mM+1)a_T) \\
&\qquad +\sigma^2((mM-1)a_T) - \sigma^2(mMa_T)\} \\
&= \frac{1}{2}\Big\{((mM+1)^{2\lambda} - (mM)^{2\lambda}) - ((mM)^{2\lambda} - (mM-1)^{2\lambda})\Big\}
\end{aligned}$$

$$\leq 2\lambda(2\lambda-1)\frac{1}{M^{2(1-\lambda)}}m^{2(\lambda-1)}, \qquad m \geq 1.$$

Since $1/2 < \lambda < 1$ and M is a big number, we have

$$0 < r_T(i,j) < \frac{2}{M^{2(1-\lambda)}}|i-j|^{2(\lambda-1)} =: \rho < 1. \tag{2.20}$$

Note that ρ may become small enough by a number M sufficiently large. In order to get the upper bound of (2.19), let us apply Lemma 5 for

$$k = k_T, \quad X_{li} = Y_T(i)/\sigma(a_T), \quad i = 1, 2, \cdots, k_T,$$
$$u = \sqrt{\frac{r-\epsilon}{r+\delta}}(1-\epsilon)\Big(2\log(\frac{T}{a_T}(\log T)^\delta)\Big)^{1/2} \quad \text{and} \quad r_{ij} = r_T(i,j).$$

Then (2.19) is less than or equal to

$$\{1-\Phi(u)\}^{k_T} + K \sum_{1\leq i<j\leq k_T} r_T(i,j)\exp\Big(-\frac{u^2}{1+r_T(i,j)}\Big). \tag{2.21}$$

Using the inequality (1.7) and condition (iv), the first term of (2.21) is bounded as below :

$$\begin{aligned}
&\{1-\Phi(u)\}^{k_T} \\
&\quad \leq \exp(-k_T\Phi(u)) \\
&\quad \leq \exp\Big(-\frac{k_T}{\sqrt{2\pi}(u+1)}e^{-u^2/2}\Big) \\
&\quad = \exp\Big(-\frac{1}{\sqrt{2\pi}}\Big[\frac{T}{Ma_T}\Big]\Big\{\sqrt{\frac{r-\epsilon}{r+\delta}}(1-\epsilon)\Big(2\log(\frac{T}{a_T}(\log T)^\delta)\Big)^{1/2}+1\Big\}^{-1} \\
&\qquad\qquad \times\Big(\frac{T}{a_T}(\log T)^\delta\Big)^{-\frac{r-\epsilon}{r+\delta}(1-\epsilon)^2}\Big) \\
&\quad \leq \exp\Big(-c\frac{T}{a_T}\Big(\frac{T}{a_T}(\log T)^\delta\Big)^{-(r-\epsilon)(1-\epsilon)/(r+\delta)}\Big) \\
&\quad \leq \exp(-c(\log T)^{\epsilon(r+1-\epsilon)})
\end{aligned} \tag{2.22}$$

for large T. Let us estimate the upper bound for the second term of (2.21). For $r \geq 2$, set

$$a = \frac{r(1-\rho-2\epsilon)-1-\rho-\epsilon(2-\epsilon)}{r(1+\rho)}$$

where ρ and ϵ are the same as in (2.19) and (2.20). Clearly, $0 < a < 1$. We split the sum in the second term of (2.21) into two parts as follows :

$$\begin{aligned}
\sum &:= K \sum_{1\le i<j\le k_T} r_T(i,j)\exp\Big(-\frac{u^2}{1+r_T(i,j)}\Big) \\
&= K \sum_{\substack{1\le i<j\le k_T\\ |i-j|\le [k_T^a]}} r_T(i,j)\exp\Big(-\frac{u^2}{1+r_T(i,j)}\Big) \\
&\qquad + K \sum_{\substack{1\le i<j\le k_T\\ |i-j|> [k_T^a]}} r_T(i,j)\exp\Big(-\frac{u^2}{1+r_T(i,j)}\Big) \\
&=: I + J.
\end{aligned}$$

Using the condition (iv) and (2.20), each sum is dominated as below :

$$\begin{aligned}
I &\le K k_T^{1+a}\exp\Big(-\frac{1}{1+\rho}\Big(\frac{r-\epsilon}{r+\delta}\Big)(1-\epsilon)^2 2\log\big(\frac{T}{a_T}(\log T)^\delta\big)\Big) \\
&\le K\Big(\frac{T}{Ma_T}\Big)^{1+a}\Big(\frac{T}{a_T}(\log T)^\delta\Big)^{-\frac{2}{1+\rho}(1-\epsilon)^2\left(\frac{r-\epsilon}{r+\delta}\right)} \\
&\le c(\log T)^{r(1+a)-(r+\delta)\left(\frac{2-2\epsilon}{1+\rho}\right)\left(\frac{r-\epsilon}{r+\delta}\right)} \\
&= c(\log T)^{r\left(1+a-\frac{2-2\epsilon}{1+\rho}\right)+\frac{\epsilon(2-2\epsilon)}{1+\rho}},
\end{aligned} \tag{2.23}$$

$$\begin{aligned}
J &\le K \sum_{\substack{1\le i<j\le k_T\\ |i-j|> [k_T^a]}} \frac{2}{M^{2(1-\lambda)}}|i-j|^{2(\lambda-1)}\exp\Big(-u^2+\frac{r_T(i,j)u^2}{1+r_T(i,j)}\Big) \\
&\le K k_T^2\frac{2}{M^{2(1-\lambda)}}k_T^{-2a(1-\lambda)}\exp\Big(-u^2+\frac{2}{M^{2(1-\lambda)}}k_T^{-2a(1-\lambda)} \\
&\qquad\qquad \times\Big(\frac{r-\epsilon}{r+\delta}\Big)(2-2\epsilon)\log(Mk_T(\log T)^\delta)\Big) \\
&\le ck_T^{2-2a(1-\lambda)}e^{-u^2} \\
&= ck_T^{2-2a(1-\lambda)}\Big(\frac{T}{a_T}(\log T)^\delta\Big)^{-(2-2\epsilon)\left(\frac{r-\epsilon}{r+\delta}\right)} \\
&\le c(\log T)^{r\left\{2-2a(1-\lambda)-(2-2\epsilon)\left(\frac{r-\epsilon}{r+\delta}\right)\right\}-\delta(2-2\epsilon)\left(\frac{r-\epsilon}{r+\delta}\right)} \\
&= c(\log T)^{r\{2-2a(1-\lambda)\}-(2-2\epsilon)(r-\epsilon)} \\
&= c(\log T)^{r\{-2a(1-\lambda)\}+\epsilon(2-2\epsilon+2r)}
\end{aligned} \tag{2.24}$$

for all large T. For $r\ge 2$, it is clear that the exponential term of (2.23) is greater than that of (2.24). So we have

$$\sum = I+J \le c(\log T)^{r\left(1+a-\frac{2-2\epsilon}{1+\rho}\right)+\frac{\epsilon(2-2\epsilon)}{1+\rho}} = c(\log T)^{-\{1+\frac{\epsilon^2}{1+\rho}\}}. \tag{2.25}$$

From (2.19), (2.21), (2.22) and (2.25), we conclude that

$$\mathbf{P}\Big\{\gamma_T \sup_{0\le t\le T-a_T}(X(t+a_T)-X(t)) \le \sqrt{\frac{r-\epsilon}{r+\delta}}(1-\epsilon)\Big\} \le c(\log T)^{-\{1+(\epsilon^2/2)\}}. \tag{2.26}$$

For $k = 1,2,\cdots$, let $T_k = \exp(k^\alpha)$ where $0 < \alpha = 1/(1+(\epsilon^2/4)) < 1$. By (2.26),

$$\mathbf{P}\Big\{\gamma_{T_k} \sup_{0\le t\le T_k-a_{T_k}}(X(t+a_{T_k})-X(t)) \le \sqrt{\frac{r-\epsilon}{r+\delta}}(1-\epsilon)\Big\} \le ck^{-\alpha\{1+(\epsilon^2/2)\}}$$

where $\alpha(1+\frac{\epsilon^2}{2}) > 1$. So, the series

$$\sum_{k=1}^{\infty}\mathbf{P}\Big\{\gamma_{T_k} \sup_{0\le t\le T_k-a_{T_k}}(X(t+a_{T_k})-X(t)) \le \sqrt{\frac{r-\epsilon}{r+\delta}}(1-\epsilon)\Big\}$$

is convergent, and hence the Borel-Cantelli lemma implies that

$$\liminf_{k\to\infty}\gamma_{T_k} \sup_{0\le t\le T_k-a_{T_k}}(X(t+a_{T_k})-X(t)) \ge \sqrt{\frac{r}{r+\delta}} \qquad \text{a.s.}$$

Again, let T be in $T_k \le T \le T_{k+1}$ for given T_k. Then it follows from Lemma 6 that for $r \ge 2$

$$\liminf_{T\to\infty}\gamma_T \sup_{0\le t\le T-a_T}(X(t+a_T)-X(t)) \ge \sqrt{\frac{r}{r+\delta}} \qquad \text{a.s.}$$

Since the function $\sqrt{r/(r+\delta)}$ on $r \ge 0$ is nonincreasing, (2.17) remains true for $r \ge 0$. From (2.13), (2.17) and (2.16), the proof of Theorem 2 is complete.

REFERENCES

(1) Berman, S. M. (1964). Limit theorems for the maximum term in stationary sequences. *Annals of Mathematical Statistics*, **35**, 502–516.

(2) Book, S.A. and Shore, T.R. (1978). On large intervals in the Csörgö-Révész theorem on increments of a Wiener process. *Zeitschrift fuer Wahrscheinlichkeitstheorie und verwandte Gebiete*, **46**, 1–11.

(3) Choi, Y.K. (1991). Erdös-Rényi laws applied to Gaussian processes. Doct. Diss., *Journal of Mathematics of Kyoto University*, **31(1)**, 191–217.

(4) Csáki, E, Csörgö, M, Lin, Z.Y. and Révész, P. (1991). On infinte series of independent Ornstein-Uhlenbeck processes. *Stochastic Processes and their Applications*, **39**, 25–44.

(5) Csörgö, M. and Révész, P. (1981). *Strong Approximations in Probability and Statistics*, Academic Press, New York.

(6) Deo, C. M. (1977). A note on increments of a Wiener process. *Technical Report,* University of Ottawa.

(7) Erdös, P. and Rényi, A. (1970). On a new law of large numbers. *Journal of Analyse Mathematics*, **23**, 103-111.

(8) Fernique, X. (1975). *Evaluations of processus Gaussiens cpmposes, Probability in Banach spaces*, Oberwalbach, Lecture Notes in Math., Springer, **526**, 67-83.

(9) Lu, C. R. (1986). Some results on increments of Gaussian processes. *Chinese Journal of Applied Probability Statistics*, **2(1)**, 59–65.

(10) Ortega, J. (1984). On the size of the increments of nonstationary Gaussian processes. *Stochastic Processes and their Applications.*, **18**, 47-56.

(11) Slepian, D. (1962). The one-sided barrier problem for Gaussian noise. *The Bell System Technical Journal*, **41**, 463–501.

(12) Strassen, V. (1964). An invariance principle for the law of iterated logarithm. *Zeitschrift fuer Wahrscheinlichkeitstheorie und verwandte Gebiete*, **3**, 211–226.